Intermittency and Nonextensivity in Turbulence and Financial Markets

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Abstract. - We present a new framework for modeling the statistical behavior of both fully developed turbulence and short-term dynamics of financial markets based on the nonextensive thermostatistics proposed by Tsallis. We also show that intermittency – strong bursts in the energy dissipation or clusters of high price volatility – and nonextensivity – anomalous scaling of usually extensive properties like entropy – are naturally linked by a single parameter q, from the nonextensive thermostatistics.

Scaling invariance plays a fundamental role in many natural phenomena and frequently emerges from some sort of underlying cascade process. A classical example is fully developed homogeneous isotropic three-dimensional turbulence, which is characterized by a cascade of kinetic energy from large forcing scales to smaller and smaller ones through a hierarchy of eddies. At the end of the cascade, the energy dissipates by viscosity, turning into heat. Recently, some authors[1−3] have studied the phenomenological relationship among financial market dynamics, scaling behavior and hydrodynamic turbulence. Particularly, Ghashghaie *et al.*[2] conjectured the existence of a temporal information cascade similar to the spatial energy cascade found in fully developed turbulence.

Traditionally, the properties of turbulent flows are studied from the statistics of velocity differences $v_r(x) = v(x) - v(x + r)$ at different scales r. As with other physical systems that depend on the dynamical evolution of a large number of nonlinearly coupled subsystems, the energy cascade in turbulence generates a spatial scaling behavior – power-law behavior with r – of the moments $\langle v_r^n \rangle$ of the probability distribution function (PDF) of v_r (the angle brackets $\langle\rangle$ denote the mean value of the enclosed quantity). For large values of the Reynolds number, which measures the ratio of nonlinear inertial forces to the linear dissipative forces within the fluid, there is a wide separation between the scale of energy input (integral scale L) and the viscous dissipation scale (Kolmogorov scale η). Though at large scales ($\sim L$) the PDFs are normally distributed, far from the integral scale they are strongly non-Gaussian and display wings fatter than expected for a normal process. This is the striking signature of the intermittency phenomenon. After publication of the Kolmogorov K62 refined similarity hypotheses[8], the problem of small scale intermittency became one of the central questions on isotropic turbulence. Over the past years several papers[9−16] have discussed intermittency and the so-called 'PDF problem'. Similar attempts[1,2,17] have been made to explain the same peculiar shape observed in the PDF of price changes $z_\tau = z(t) - z(t + \tau)$ at small time intervals.

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Based on the scaling properties of multifractals, Tsallis[4–7] has proposed a generalization of Boltzmann-Gibbs thermostatistics by introducing a family of generalized nonextensive entropy functionals $S_q[p]$ with a single parameter q. These functionals reduce to the classical, extensive Boltzmann-Gibbs form as $q \to 1$. Optimizing $S_q[p]$ subject to appropriate constraints[6], we obtain the distribution

$$p_q(x) = [1 - \beta(1-q)x^2]^{1/(1-q)}/Z_q \quad . \tag{1}$$

The appropriate normalization factor, for $1 < q < 3$, is given by

$$Z_q \equiv \left[\frac{\beta(q-1)}{\pi}\right]^{1/2} \frac{\Gamma(1/(q-1))}{\Gamma((3-q)/2(q-1))} \quad .$$

In the limit of $q \to 1$, we recover the Gaussian distribution.

The above distribution, we claim, provides the simplest and most accurate model for handling the PDF problem. To show this, we stay in the context of fully developed turbulence ($x \equiv v_r$). From equation (1), we can easily obtain the second moment

$$\langle v_r^2 \rangle = \frac{1}{\beta(5-3q)} \quad , \tag{2}$$

and the flatness coefficient (kurtosis)

$$K_r = \frac{\langle v_r^4 \rangle}{\langle v_r^2 \rangle^2} = \frac{3\,(5-3q)}{(7-5q)} \quad . \tag{3}$$

We remark that the flatness coefficient, which is directly related to the occurrence of intermittency, is solely determined by the parameter q. Also, we note that the positiveness of K_r sets an upper bound on the value of q, namely $q < 7/5$. This bound coincides with the one obtained by Boghossian[18] through a q-generalization of Navier-Stokes equations. Moreover, this limit implies that the second moment of distribution (1) will always remain finite, which is empirically expected from the phenomena here analyzed.

At this point, if we assume[2,9,10] a scaling of the moments $\langle v_r^n \rangle$ of v_r as r^{ζ_n}, the variation with r of the PDF of the velocity differences and of its related moments can be completely determined. Under the assumptions of Kolmogorov log-normal model[8], $\zeta_n = n/3 - \frac{1}{18}\mu n(n-3)$, where μ is the intermittency exponent. Then, we immediately obtain the functional forms of the flatness coefficient and the parameter q, respectively

$$K_r = K_L(r/L)^\alpha \tag{4}$$

and

$$q = \frac{5 - 7(r/L)^\alpha}{3 - 5(r/L)^\alpha} \quad , \tag{5}$$

with $K_L = 3$, the expected value for a Gaussian process, and $\alpha = -4\mu/9$. The correspondent expression for β can be derived similarly from equation (2). However, to account for the well known asymmetry of the velocity distributions we may consider $\beta = \beta^+$, for $v_r \geq 0$, and $\beta = \beta^-$, for $v_r < 0$. In this case, both the second and third moments of the modified PDF shall be used to determine β^- and β^+. Equations (3) and (5) remain unchanged, as far as the asymmetry of the PDF is small.

We checked our model with turbulence statistics data taken from reference 2, provided by Chabaud *et al.*[13]. Firstly, we adjusted by least-squares to the experimental data, the parameters of our PDF, considering both the symmetric and asymmetric forms of equation (1). Then, we compared the estimated values of q with the predictions of equation (5), for a intermittency coefficient of $\mu = 0.25$ (the best estimate currently available is[12] 0.25 ± 0.05). The results are displayed in Figs. 1a and 2. A good agreement is observed through all spatial scales and for all orders of normalized velocity differences. Note that the solid line in Fig. 2 is not simply a fit to the data, since μ is obtained independently. We also computed value of q at the Kolmogorov scale and obtained $q_\eta \simeq 9/7$.

The same approach adopted in turbulence can be straightforwardly applied (with $x \equiv z_\tau$) to model the statistics of price differences in financial markets, as far as the relevant parameters at the integral time scale – time span for which a convergence to a Gaussian process is found – are available. We tested our model with price changes data taken from reference 2, provided by Olsen & Associates. The results are displayed in Figs. 1b and 2. Since we do not have an independent estimate of the intermittency coefficient of the information cascade, we also adjusted equation (5) to the data and obtained $\alpha = -0.17$ and $\tau_L \simeq 2.2$ days, the corresponding integral scale of the process. We observe that the proposed model reproduces with good accuracy the statistics of price differences over all temporal scales. However, we remark that the integral scale value of 2.2 days strongly disagrees with other estimates[1] of τ_L (roughly 1 month).

Nonextensivity, a matter of speculation in some areas[19], is an essential feature of Tsallis generalized thermostatistics. If we suppose a scenario of a cascade of bifurcations with n levels, and scale the generalized entropy S_q, averaged over a volume of size $V = \eta^3$ and normalized by Boltzmann constant, we have at the first level

$$S_q(2V) = 2S_q(V) + (1-q)S_q^2(V) \tag{6}$$

and at the top of the cascade

$$S_q(2^nV) = 2S_q(2^{n-1}V) + (1-q)S_q^2(2^{n-1}V) \simeq 2^nS_q(V) + (1-q)2^{n-1}S_q(V) \quad , \tag{7}$$

where the higher order terms in S_q have been discharged. Cascade processes are also described in terms of fractal or multifractals models[11,20−23]. Within these frameworks, in high Reynolds number turbulence, the energy dissipation is not uniformly distributed within the fluid but rather concentrated on subsets of non-integer fractal D_F dimension. This picture leads to a scaling behavior with dimensionality not equal to that of the embedding space. In this case, if we consider the cascade of bifurcations described above, we find

$$S_q(2^nV) = 2^{nD_F/3}S_q(V) \quad . \tag{8}$$

It follows immediately from equations (7) and (8) that

$$D_F \simeq \frac{3}{n}\left[\frac{log(3-q)}{log(2)} + n - 1\right] \quad . \tag{9}$$

Note that the parameter q, through equation (9), offers a quantitative picture of the transition from small-scale intermittent, nonextensive, fractal behavior to large-scale Gaussian, extensive homogeneity. For higher values of n and $q = 1$, at the top of the cascade, we have $D_F = 3$. At the bottom of the cascade ($n = 1$), using the value $q_\eta \simeq 9/7$, estimated previously, we get $D_F \simeq 2.33$, a good approximation of the so-called 'magic' value of 2.35, often measured in different experimental contexts[24–27]. This result and the divergence of the correlation function of the energy dissipation in the fluid[9]

$$\langle \epsilon(x)\epsilon(x+r)\rangle \sim \frac{\langle v_r^6\rangle}{r^2} = \frac{1}{r^2}\left[\frac{15\beta^{-2}}{(5-3q)(7-5q)(9-7q)}\right] \tag{10}$$

suggest a more stringent bound on q, namely $q < 9/7$. If true, this new bound immediately implies that, in fully developed turbulence, relation $(\eta/L)^{\alpha} = 2$ is invariant, regardless the Reynolds number.

Though only qualitatively, the above picture may be also applied to the information cascade. However, since there is nothing equivalent to viscous damping in the dynamics of speculative markets, the information cascade depth is only limited by the minimum time necessary to perform a trading transaction.

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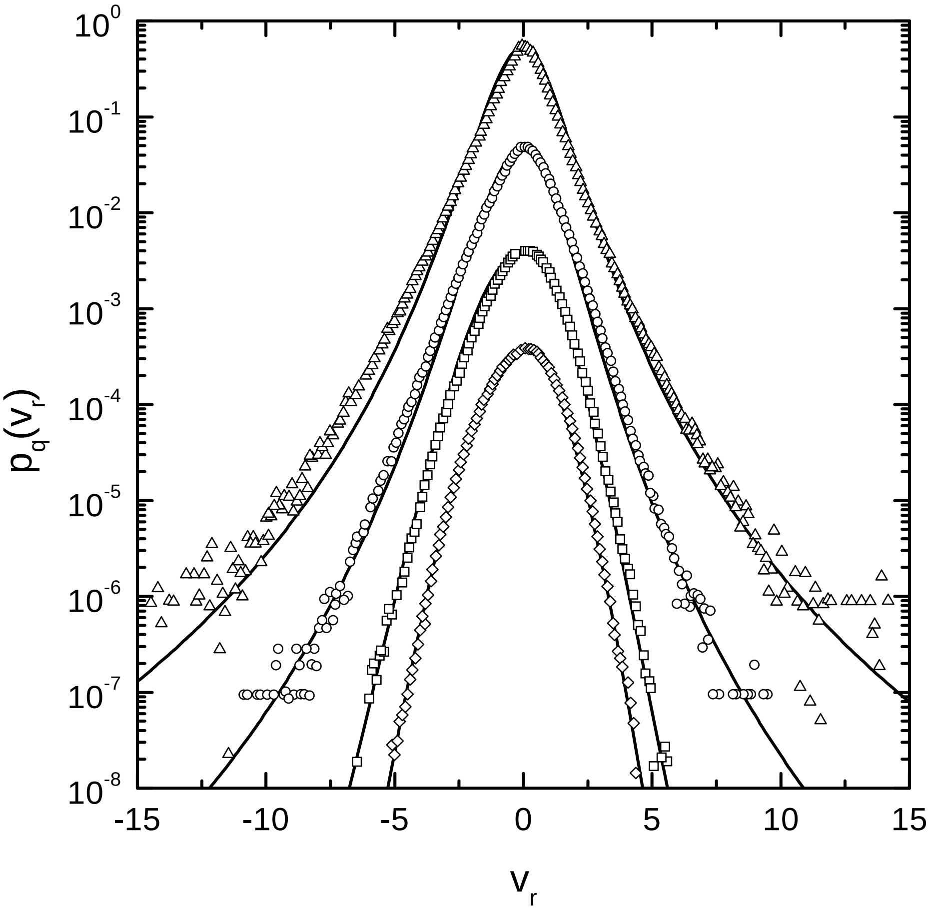
Captions

Figure 1 (a) Data points: standardized probability distribution $p_q(v_r)$ of velocity differences $v_r(x) = v(x) - v(x+r)$ for spatial scales $r = 0.0073L$, $0.0407L$, $0.3036L$, $0.7150L$, with $L/\eta = 454$ and L and η being, respectively, the integral and Kolmogorov scales (see text); data taken from ref. 2, provided by Chabaud *et al.*[13]; Solid lines: least-squares fits of modified PDF (1); from top to bottom: $q = 1.26$, 1.20, 1.11, 1.08; $\beta^- = 0.69$, 0.66, 0.55, 0.62; $\beta^+ = 0.88$, 0.82, 0.76, 0.70 (for better visibility the curves have been vertically shifted with respect to each other).

Figure 1 (b) Data points: standardized probability distribution $p_q(z_\tau)$ of price differences $z_\tau = z(t) - z(t+\tau)$ for temporal scales $\tau = 0.0035\tau_L$, $0.0276\tau_L$, $0.2210\tau_L$, $0.8838\tau_L$, with $\tau_L = 186265$ s being the integral scale (see text); data taken from ref. 2, provided by Olsen & Associates; Solid lines: least-squares fits of modified PDF (1); from top to bottom: $q = 1.35$, 1.26, 1.16, 1.11; $\beta^- = 1.12$, 0.83, 0.75, 0.75; $\beta^+ = 0.98$, 0.72, 0.61, 0.77.(for better visibility the curves have been vertically shifted with respect to each other).

Figure 2 Dependence of the parameter q on normalized spatial (r/L) and temporal scales (τ/τ_L); turbulence data: solid line (model), open squares (data, asymmetric adjust), solid squares (data, symmetric adjust); financial data: dotted line (model), open triangles (data, asymmetric adjust), solid triangles (data, symmetric adjust).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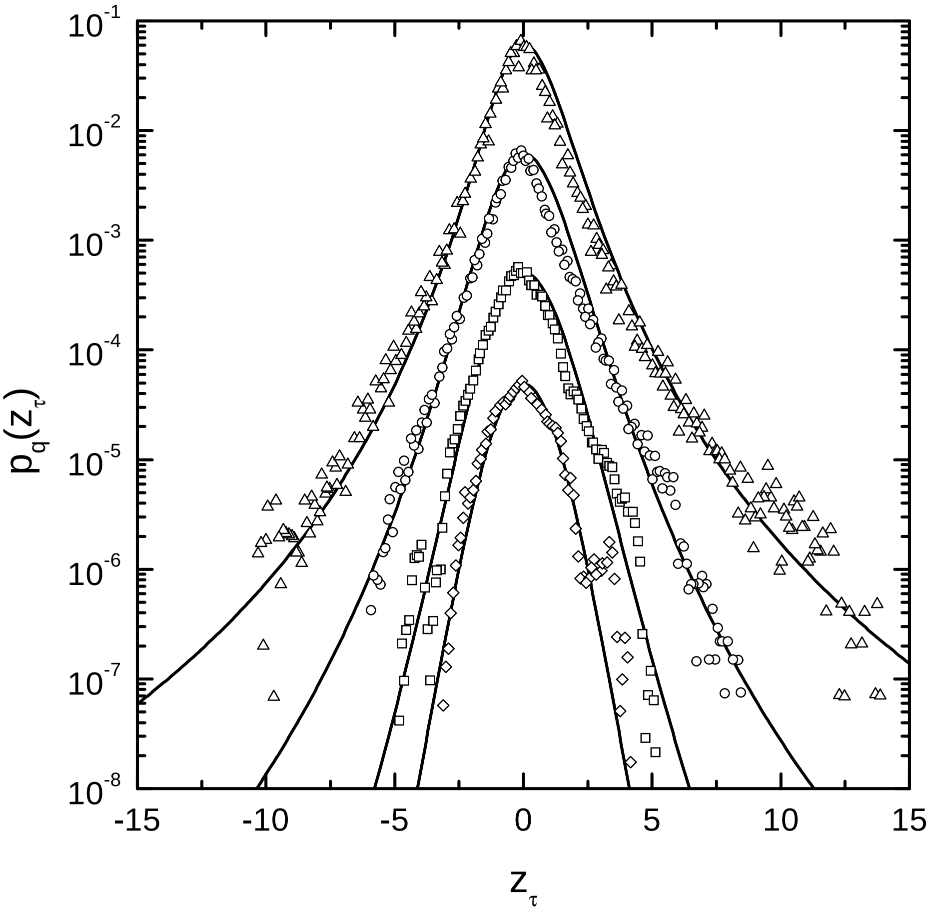
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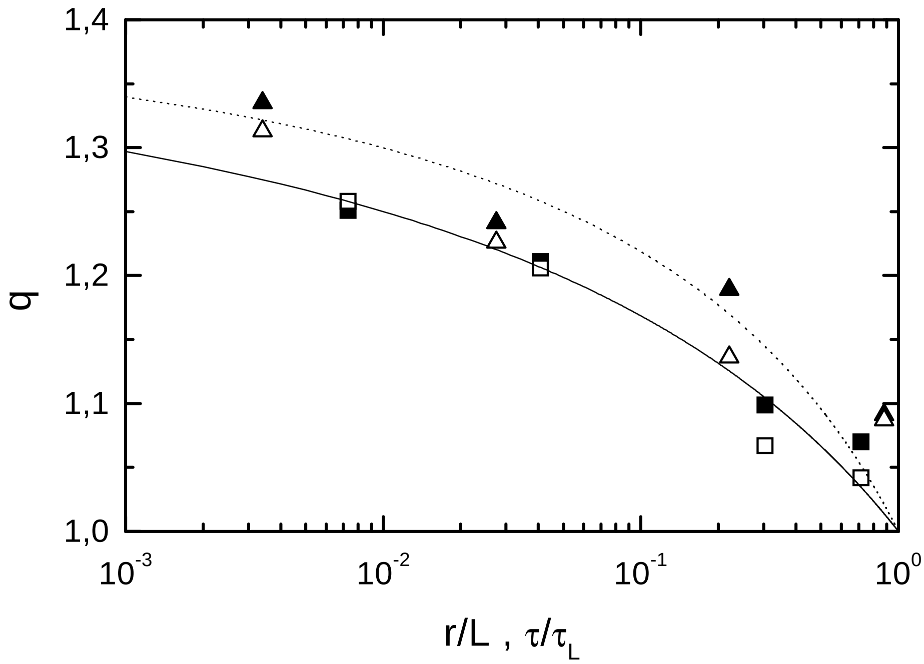
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