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SURFACE INTERPOLATION USING WEIGHTED SPLINES

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14. Abstract/Notes

The surface generating method proposed by Coons is extensively used in designing automobile bodies, ship hulls and other complex forms which are impossible to be analytically described. In order to describe the "cages", which basically provide these surfaces, Bézier or B-Spline methods are traditionally used. These methods, however, have a disadvantage, since the control points do not belong to the generated curves and this creates difficulties in modifying the design. The present work proposes, for generating the "cages", a method known as Weighted Splines, of great computational efficiency and without the restriction regarding the control points, since, in this method, they belong to the generated curves. The method is also shown to be a particular case of Coons' ideas. In addition to this, a method to allow the setting of derivatives of the curve at the desired points is also presented, in which even a discontinuity may be generated when necessary.

15. Remarks *This paper submitted for consideration to Eurographics'85 - European Computer Graphics Conference and Exhibition to be held in Nice, France, during 9-13 September, 1985.*

1. COONS' METHOD

1.1 - SURFACE EQUATION

A known method for interpolating surfaces was developed at the Massachusetts Institute of Technology by Coons (1967). Its main idea is the following:

Assume a "patch", as shown in Figure 1, defined by its four boundaries. These boundaries may be defined parametrically or by points. The only restriction imposed is that they must be closed at the four corners. These four boundaries, defined in natural coordinates - [0,1] - are given by functions g(i,j).

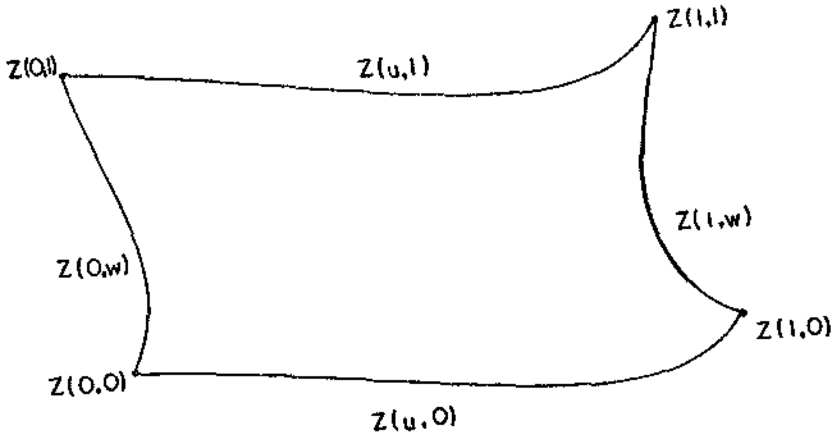


Fig. 1 - Surface "patch".

It can be verified that the surface equation below contains the four curves g which define the boundaries and that the equation is defined by them.

$$Z(u,w) = \sum_{i=0}^{1} g(i,w)\, F_i(u) + \sum_{i=0}^{1} g(u,i)\, F_i(w) + $$

$$- \sum_{i=0}^{1} \sum_{j=0}^{1} g(i,j)\, F_i(u)\, F_j(w) \qquad (1.1)$$

The functions F_i (i=0,1) are known as "Blending" Functions and the following conditions are stipulated so that the boundaries belong to the surface:

$$F_i(j) = \delta_{i,j} \tag{1.2}$$

One can say that the surface is generated by a gradual transition from one boundary to the other and that these two boundary shapes are "blended" together by virtue of the "blending" functions F_i. The subtracting term is necessary because the four corner points g(i,j) are common to the boundaries g, and this makes them appear twice in the equation.

A further stipulation is that F_i be continuous and monotonic over the interval [0,1], so that the generated surface will also be a continuous one, without presenting oscillations in the interval.

1.2 - SURFACE CONTINUITY

Consider two "patches", A and B, with a common boundary as shown in Figure 2.

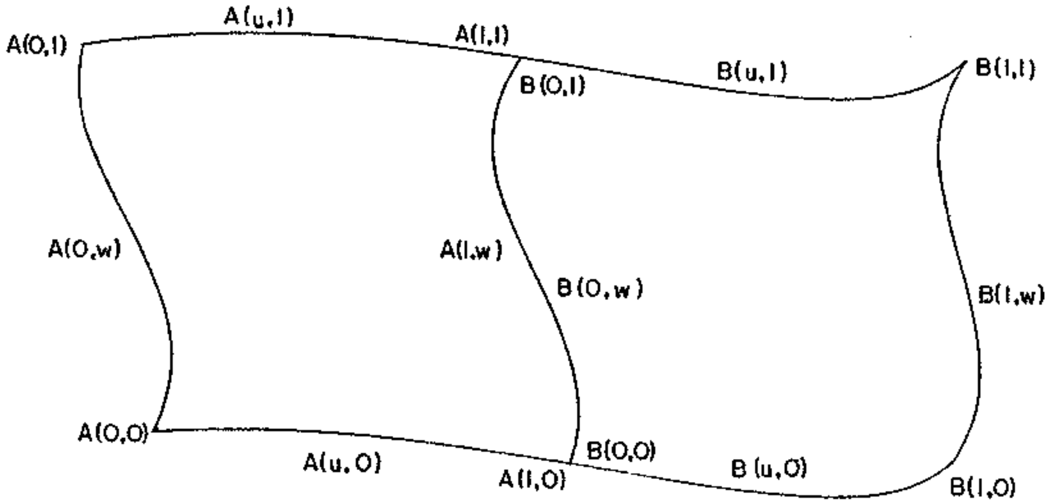


Fig. 2 - Two "patches" with a common boundary.

For patch A the boundary is A(1,w); for "patch" B it is B(0,w) and the vectors of coordinates are equal to:

$$A(1,w) = B(0,w).$$

Then, the two "patches" will be continuous across their common boundary. They will, however, be discontinuous in slope across the boundary and it is necessary to investigate this and make some corrections that will eliminate this slope discontinuity.

1.3 - BOUNDARY SLOPE CONTINUITY

Stipulating the following conditions:

$$F_i'(j) = 0 \quad i,j = 0,1 \tag{1.3}$$

for the derivate of Equation 1.1, the derivatives of Z along the common boundary of Figure 2 are described by the following expression (given in terms of coordinates of the left "patch"):

$$Z(1,w)_u = \sum_{i=0}^{1} g(1,i)_u \, F_i(w) \tag{1.4}$$

where u is the coordinate in terms of which Equation 1.1 was derived.

Since Equation 1.1 is symmetric, the same can be shown for any boundary.

1.4 - BOUNDARY CURVATURE CONTINUITY

In an analogous way, it is shown that stipulating the following conditions.

$$F_i''(j) = 0 \quad i,j = 0,1 \tag{1.5}$$

for the second derivative of Equation 1.1, the second derivative of Z along the common boundary is described by the following expression:

$$Z(1,w)_{uu} = \sum_{i=0}^{1} g(1,i)_{uu}\, F_i(w). \qquad (1.6)$$

It is easy to see that in this manner one may go further to any level of derivative continuity along contiguous boundaries.

2. SLOPE CORRECTION SURFACE

The necessity of specifying the values of derivatives in a surface to be interpolated is very common in many applications.

The surface equation described above has a definite intrinsic slope along the boundaries, the variation of which are strictly prescribed by a simple formula in terms of the derivatives at the corner points (Equation 1.4).

In order to allow the setting of derivative values along the boundaries, one can proceed in the same manner as before, that is, developing a new surface, known as "correction surface", which may be added to the original one (Equation 1.1), thus producing the desired derivative value along the boundaries, whithout changing the surface and the curvature, if necessary, along the boundaries.

Consider the surface

$$\alpha(u,w) = \sum_{i=0}^{1} f(i,w)_u\, G_i(u) + \sum_{j=0}^{1} f(u,j)_w\, G_j(w) + \\ - \sum_{i=0}^{1} \sum_{j=0}^{1} f(i,j)_{uw}\, G_i(u)\, G_j(w). \qquad (2.1)$$

Proceeding exactly in an analogous way, the following conditions are imposed in order to obtain the necessary characteristics:

$$\left.\begin{aligned} G_i(j) &= 0 \\ G_i'(j) &= \delta_{i,j} \\ G_i''(i) &= 0 \end{aligned}\right\} \qquad i,j = 0,1. \tag{2.2}$$

One can notice that the new surface $\alpha(u,w)$ satisfies the necessary restriction of changing the slope along the boundaries without changing the continuities of surface and curvature.

As this correction surface is valid inside the "patch", it can also be used to break the continuity of the slope which Equation 1.1 normally enforces.

One should note that the function f_u, f_w and f_{uw} of Equation 2.1 are differences between the new values imposed to the derivatives and intrinsic slopes in the surface equation. It can also be seen that derivatives of any order may be corrected.

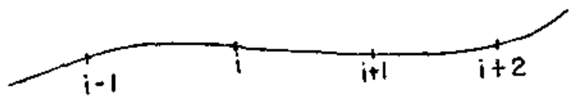
3. BOUNDARY INTERPOLATION

The surfaces are obtained joining the "patches" using Coons' method. In order to generate the "patch", one has to define the four boundaries from the control points. There are many ways to obtain these boundaries - polynomial interpolation like Langrange, Hermite, etc., Cubic Splines, (Forsythe et alii, 1977), Bézier, B-Spline, (Newman and Sproull, 1979), etc. The first three methods have two disadvantages - the great computer effort and the lack of guarantee that a smooth curve can be described. The last two are very fast, provide smooth curves, but the control points do not belong to the generated curve except for the extreme ones.

An interpolating method known as Weighted Spline (Costa, 1980) is proposed, which is very fast, smooth, and has the advantage that the control points belong to the generated curve. This method is shown to be a particular case of Coons' ideas, when his method is reduced to two dimensions.

3.1 - WEIGHTED SPLINE

Consider a sequence of points 1, 2, ..., i-1, i, i+1, i+2, ..., n-1, n through which one wishes to interpolate a curve or boundary:



In order to obtain the segment (i, i+1), the points (i-1) and (i+2) are also considered because they provide a general trend of the curve before and after the segment. Two parabolas, h_0 and h_1, are used to interpolate through i-1, i and i+1, and i, i+1 and i+2, respectively.

In the same way as in the three-dimensional case, the segment (i, i+1) is represented by a suitable "blend" of the parabolas of the form.

$$P(x) = \sum_{i=0}^{1} \lambda_i(x)\, h_i(x), \qquad (3.1)$$

where λ_i are the "blending" functions.

In order to guarantee the continuity of the first and second derivatives, a polynomial to interpolate between points i and i+1 has at least to be of fifth degree. Since the parabolas are of second degree, third degree "blending" functions will be considered, which satisfy the following conditions:

$$\left.\begin{array}{l} \lambda_i(j) = \delta_{i,j} \\ \lambda_i'(j) = 0 \\ \lambda_0''(i) + \lambda_1''(0) = 0 \end{array}\right\} \qquad i,j = 0,1 \tag{3.2}$$

Although it can be noticed that there are more equations than unknowns, one can find that the following two solutions satisfy all the conditions in Equation 3.2:

$$\lambda_0\ (x) = 2x^3 - 3x^2 + 1$$

$$\lambda_1\ (x) = 1 - \lambda_0\ (x) = -2x^3 + 3x^2 \tag{3.3}$$

The above "blending" functions guarantee the continuity of the curve, the first and second derivatives satisfying the conditions in Equation 3.2. They do not satisfy simultaneously conditions in Equation 1.2, 1.3 and 1.5; they satisfy only conditions in Equations 1.2 and 1.3. These "blending" functions may be used for the three-dimensional case if only first derivative continuity is necessary.

3.2 - SLOPE CORRECTION FOR CURVES

As in the case of the slope correction surface, one can specify slopes for the end points of the segment. This would yield the following equation:

$$C(x) = \gamma_i(x)\ D_i, \tag{3.4}$$

where $\gamma_i(x)$ are the "blending" functions and D_i are the derivatives at the end points of the segment. The conditions are:

$$\left.\begin{aligned} \gamma_i(j) &= 0 \\ \gamma_i'(j) &= \delta_{i,j} \\ \gamma_i''(j) &= 0 \end{aligned}\right\} \qquad i,j = 0,1.$$

Considering only the first two conditions, one can obtain:

$$\gamma_0(x) = x^3 - 2x^2 + x,$$

$$\gamma_1(x) = x^3 - x^2.$$

Figure 3 shows an example, using the weighted spline method, which passes through five data points. The same curve is shown in Figure 4, with new derivatives at the third point. The Figure 5 shows in detail the region where the derivatives have been corrected; and a complete surface is shown in Figure 6.

4. CONCLUSIONS

The particular case of Coons' method for the bidimensional case shows many advantages over the traditional methods: the small computational effort, which increases linearly with the number of data points; the smoothness of the interpolated segment is guaranteed; the curves pass through all data points; the continuity of any order is guaranteed (if one is prepared to pay the computational cost); and it allows one to have a complete control over the slopes, which makes possible to specify them, and also to produce discontinuities where necessary.

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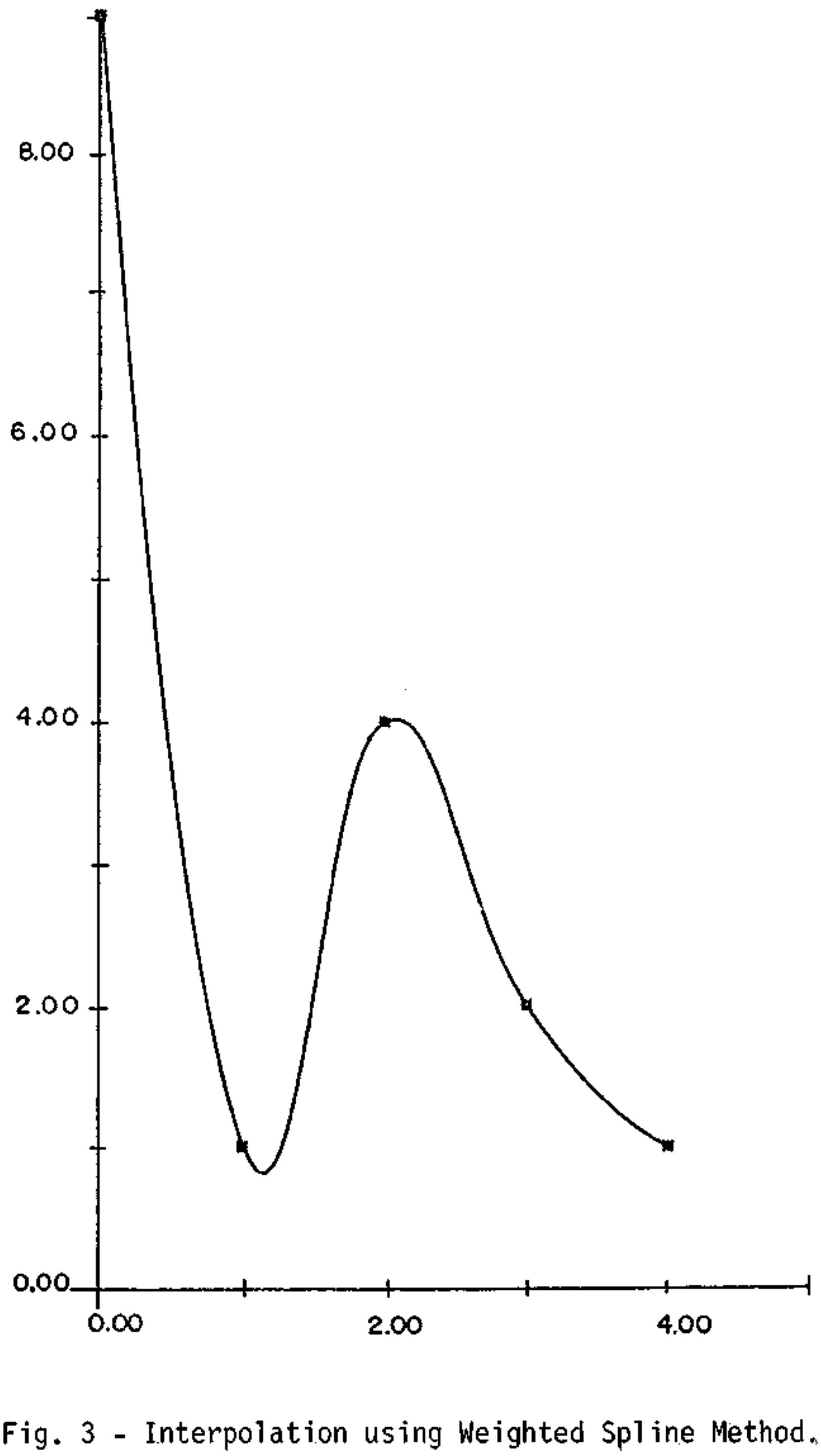

Fig. 3 - Interpolation using Weighted Spline Method.

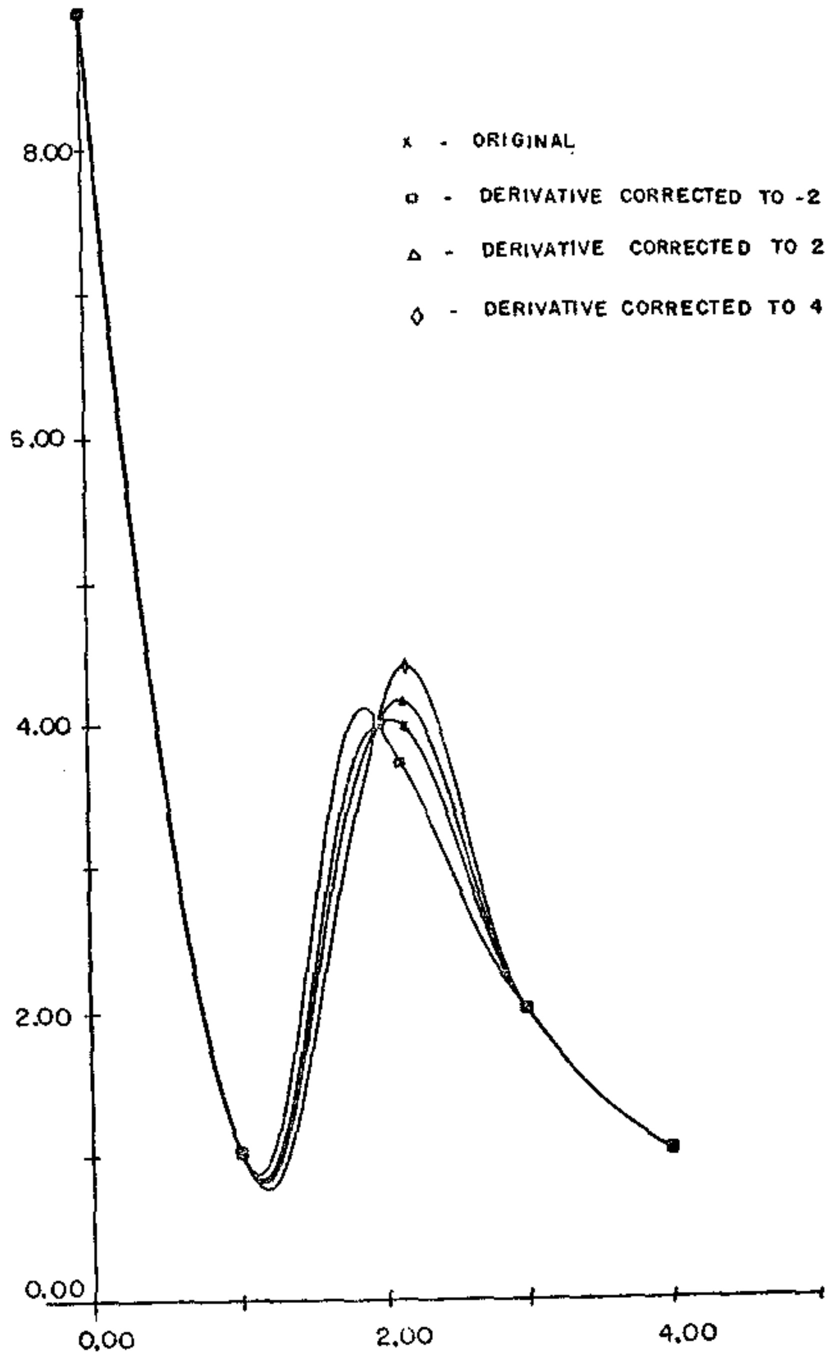

Fig. 4 - Original curve with new forms after slope correction.

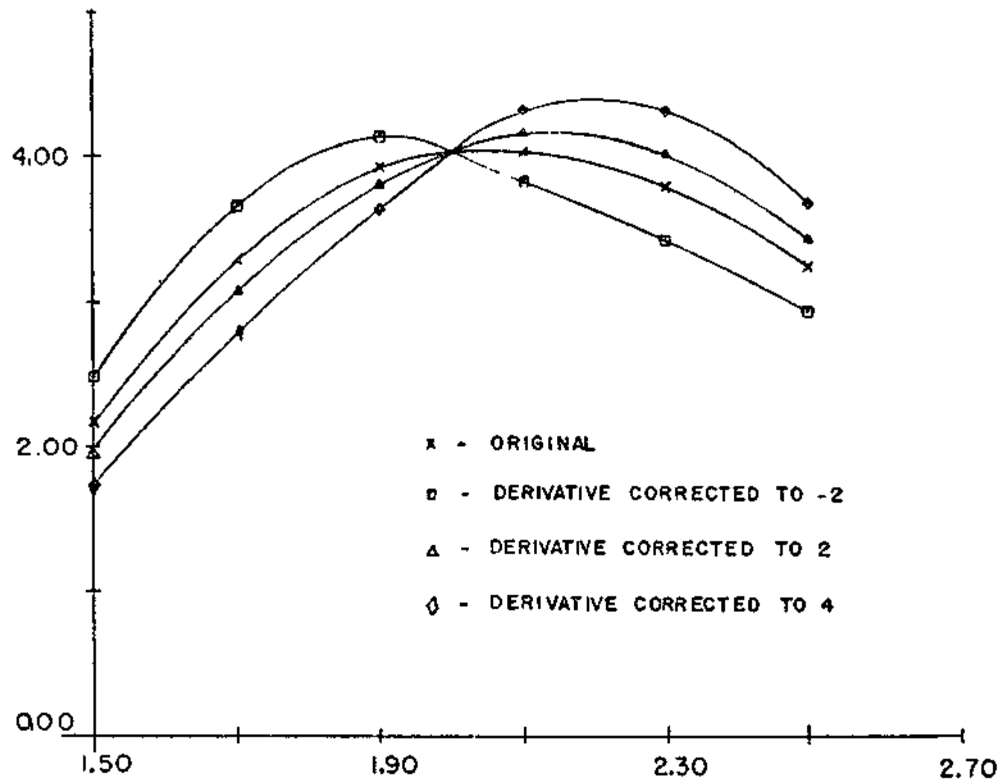

Fig. 5 - Details of the region where the derivatives have been corrected.

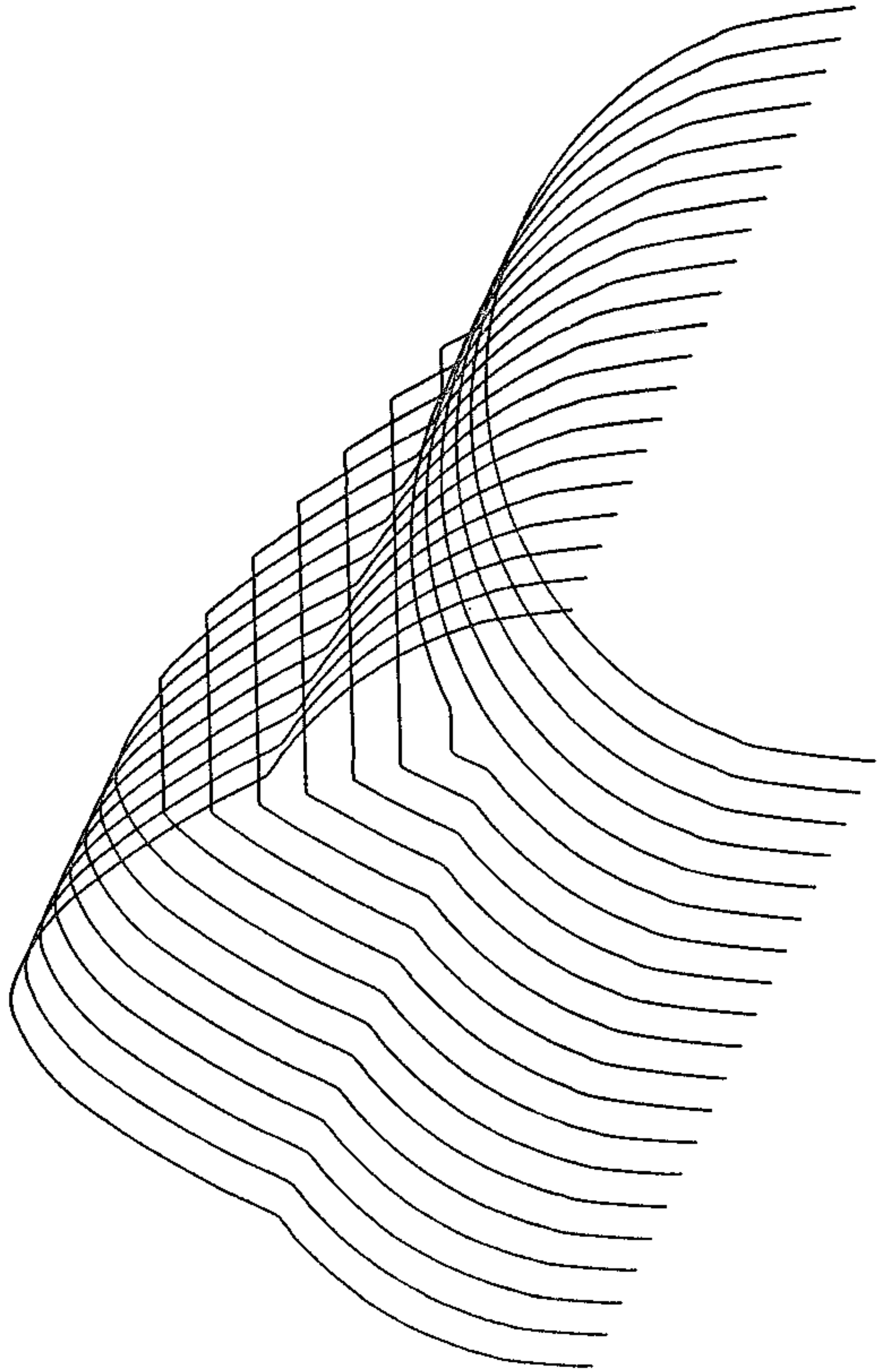

Fig. 6 - Surface generated using Weighted Spline Interpolation for the boundaries.