THREE-DIMENSIONAL TRANSIENT HEAT DIFFUSION IN A MULTILAYERED ORTHOTROPIC PLATE USING A FINITE ANALYTIC NUMERICAL METHOD

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ABSTRACT

A finite analytic (FA) alternating direction implicit (ADI) numerical scheme is presented and applied to the solution of three-dimensional transient heat conduction problems in multilayered composite plates. The basic idea of the FA method is the incorporation of a local analytic solution of the governing equation in the numerical solution of the boundary-value problem. The FA numerical solution is applied to two sample problems. Numerical and experimental results show that the FA method is accurate and well suited for simulating the transient temperature response of composite samples.

1. INTRODUCTION

During the past decades, composite materials have found many practical applications in various industrial fields (aerospace, nuclear, micro-electronics, etc.). The increasing requirements for assuring the integrity of composite hardware have called for appropriate and efficient non-destructive evaluation (NDE) methods. Recently, pulsed photothermal NDE techniques have successfully been applied to coatings (Aamodt *et al.* 1990), thin films (Leung and Tam 1988) and layered materials (Balageas *et al.* 1991). In such techniques, the surface temperature of the inspected sample, after being irradiated with a short heat pulse, is monitored by an infrared detector. While the absorbed energy diffuses as heat through the material, the subsurface features and defects are revealed in the form of variations in the sample surface temperature.

Due to the rapid development of the composite materials technology, the study of heat conduction in multilayered media has been handled by many authors. Apart from purely numerical approaches (Tamma and Yurko 1989; Muzzio and Solaini 1987), some analytical techniques are also available (Ozisik 1980; Mikhailov *et al.* 1983) . Nevertheless, the simulation of the thermal behavior of a multidimensional composite structure with a large number of layers, during a pulsed photothermal NDE experiment, remains a difficult task for most analytical and numerical methods.

On the one hand, the analytical solutions are usually very complex and cumbersome to implement. This is due to the necessity of solving the corresponding Sturm-Liouville eigenvalue problem, which is not of the conventional type because of the discontinuity of the coefficient functions, and to the slow convergence of the costumary solutions expressed in terms of double or triple infinite series. On the other hand, the standard numerical methods generally require a fine grid and have a tendency to become time consuming when dealing with problems with local high temperature gradients and when a high accuracy level is sought.

In the present study, the problem of the transient temperature response of a three-dimensional composite plate consisting of parallel layers of orthotropic homogeneous materials, in imperfect thermal contact, is solved by the finite analytic (FA) method using an alternating direction implicit (ADI) calculation scheme. Our primary purpose is to provide an accurate and efficient numerical procedure to predict the transient temperature response of a composite sample during a pulsed photothermal NDE experiment.

In the following sections, the formulation of the FA method is detailed and applied to two sample problems. Comparison with both simulated and experimental results demonstrate the effectiveness of the numerical technique.

2. MATHEMATICAL FORMULATION

It is assumed that the composite plate consists of I parallel layers of thickness $\delta x_i = x_{i+1} - x_i$, as shown in Fig.1. Each layer is homogeneous, orthotropic and may have thermal properties that are different from those of the adjacent layers. The surface film coefficient h_i at the interfaces is allowed to vary in the y and z-directions. For $t > 0$, heat is transfered from the outer boundaries according to boudary conditions of third kind. There is no heat generation in the media.

The mathematical formulation of this heat diffusion problem governing the temperature distribution in the ith section, $x_i \leq x \leq x_{i+1}$, $i = 1, 2, \ldots, I$, is given as :

$$k_{xi}\frac{\partial^2 T_i}{\partial x^2} + k_{yi}\frac{\partial^2 T_i}{\partial y^2} + k_{zi}\frac{\partial^2 T_i}{\partial z^2} = \rho_i C_i \frac{\partial T_i}{\partial t} \quad , \qquad (1)$$

where $T_i(x, y, z, t)$ is the temperature of the ith section, ρ_i is the density, C_i is the specific heat and k_{xi}, k_{yi}, k_{zi} are the thermal conductivities in the x, y and z-directions.

The composite structure is subjected to the following boundary conditions :

$$-k_{x1}\frac{\partial T_1}{\partial x} + h_1 T_1 = f_0(t) \quad , \qquad (2)$$

$$k_{xI}\frac{\partial T_I}{\partial x} + h_{I+1} T_I = f_{I+1}(t) \quad , \qquad (3)$$

at $x = x_1 = 0$ and $x = x_{I+1} = L_x$;

$$-k_{yi}\frac{\partial T_i}{\partial y} + h_y T_i = f_y'(t) \quad , \qquad (4)$$

$$k_{yi}\frac{\partial T_i}{\partial y} + h_y T_i = f_y(t) \quad , \qquad (5)$$

at $y = 0$ and $y = L_y$, for $x_i \leq x \leq x_{i+1}$; and

$$-k_{zi}\frac{\partial T_i}{\partial z} + h_z T_i = f_z(t) \quad , \qquad (6)$$

$$k_{zi}\frac{\partial T_i}{\partial z} + h_z T_i = f_z(t) \quad , \qquad (7)$$

at $z = 0$ and $z = L_z$, for $x_i \leq x \leq x_{i+1}$.

At the interfaces, the media are subjected to the following conditions :

$$-k_{xi}\frac{\partial T_i}{\partial x} = h_{i+1}(T_i - T_{i+1}) \quad , \qquad (8)$$

$$k_{xi}\frac{\partial T_i}{\partial x} = k_{xi+1}\frac{\partial T_{i+1}}{\partial x} \quad , \qquad (9)$$

with $x = x_{i+1}$, where $h_{i+1}(y, z)$ is the surface film coefficient. Non-capacitive internal defects are also represented by this kind of boundary conditions.

For $t = 0$, the initial condition is expressed by

$$T_i(x, y, z, t) = F_i(x, y, z) \quad . \qquad (10)$$

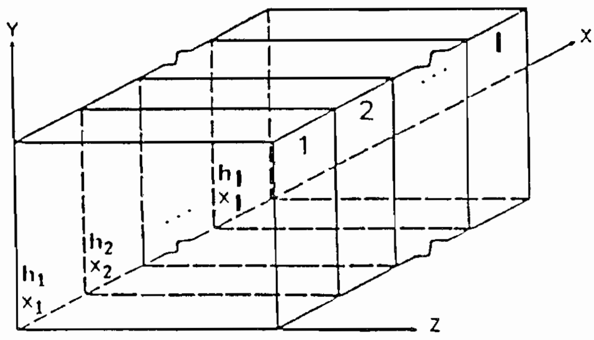

Figure 1: Multilayered composite plate.

3. DESCRIPTION OF THE FA METHOD

The FA method has been developed by Chen and his collaborators (1984,1988) to remedy the difficulties experienced in the numerical solution of fluid flow and heat transfer problems. It has recently been extended to the solution of transient heat conduction problems on 1D multilayered composite slabs with linear and non-linear boundary conditions by Ramos and Giovannini (1992).

The basic idea of the FA method is to incorporate a local analytical solution in the numerical solution of the governing partial differential equations, thus reducing the truncation error in the finite difference approximation and eliminating the use of an approximation function in the finite element method. The FA method decomposes the computational domain into a number of small discrete elements in which local analytic solutions are obtained due to locally simple geometry, equation or boundary condition. The FA method has some attractive features. It is unconditionally stable and allows one to use a coarser computational mesh without loss of accuracy.

The ADI FA solution strategy adopted in the present work comprises the following major steps :

1. Splitting of Eq.(1) into a triplet of one-dimensional equations, as follows

$$\frac{\partial^2 T_i}{\partial x^2} = \frac{1}{\alpha_{xi}}\frac{\partial T_i}{\partial t} \quad , \qquad (11)$$

$$\frac{\partial^2 T_i}{\partial y^2} = \frac{1}{\alpha_{yi}}\frac{\partial T_i}{\partial t} \qquad (12)$$

and

$$\frac{\partial^2 T_i}{\partial z^2} = \frac{1}{\alpha_{zi}}\frac{\partial T_i}{\partial t} \quad , \qquad (13)$$

where α_{xi}, α_{yi} and α_{zi} are the thermal diffusivities in the x, y and z-directions, respectively.

2. Discretisation of the computational domain in the x, y and z-directions into small one-dimensional elements $R^{(j)}$, $R^{(k)}$ and $R^{(l)}$, respectively. Any interior element $R^{(j)}$, delimited by the nodes $j-1$ and $j+1$, shall be completely overlapped by subregions $R^{(j-1)}$ and $R^{(j+1)}$.

3. On all the interior nodes, boundary conditions of the first type are given and allowed to vary linearly during a time step $\Delta t = t^m - t^{m-1}$.

4. Derivation of the local analytic solutions. The external boundary conditions are automaticaly enforced by incorporating into the numerical procedure the local analytic solutions for the outer elements (in the x-direction, $R^{(1)}$ and $R^{(J)}$).

5. Assembling and overlapping of all local analytic solutions in each orthogonal direction.

The resulting set of tridiagonal algebraic equations, solved sequentially in the x, y and z-directions, for each j, k and l grid lines, provides the temperature distribution T^m_{jkl} throughout the computational domain.

4. NUMERICAL SOLUTION

Let $T_j(\xi, t)$ be the solution of Eq.(1) in the element $R^{(j)}$, $\xi_{j-1} \le \xi \le \xi_{j+1}$, where ξ is a local space coordinate. Evaluating $T_j(\xi, t)$, at $\xi = \xi_j$ and $t = t^m$, as a function of the initial and boundary conditions of the local problem, expressed in terms of the element nodal values, namely [1]

$$F_j^{m-1}(\xi) = f(T_{j-1}^{m-1}, T_j^{m-1}, T_{j+1}^{m-1}, \xi) \quad , \qquad (14)$$

$$T_{j-1}(t) = f(T_{j-1}^{m-1}, T_{j-1}^{m}, t) \qquad (15)$$

and

$$T_{j+1}(t) = f(T_{j+1}^{m-1}, T_{j+1}^{m}, t) \quad , \qquad (16)$$

one can obtain a six-point formula for the temperature T_j^m in the form

$$T_j^m = C_{j-1} T_{j-1}^m + C_{j+1} T_{j+1}^m + P_j^{m-1} \quad , \qquad (17)$$

where the implicit coefficient terms are given by

$$C_{j-1} = \frac{1}{\Delta t} \sum_{n=1}^{N} \frac{e^{-\lambda_n^2 \Delta t}}{N_{j,n}} \psi_{j,n}(\xi_j) \left(k_{xj} \frac{d\psi_{j,n}}{d\xi} \right)_{\xi=\xi_{j-1}} \int_0^{\Delta t} e^{\lambda_n^2 t'} t' \, dt' \quad , \qquad (18)$$

[1] for clarity, the subscripts k and l have been suppressed

$$C_{j+1} = -\frac{1}{\Delta t} \sum_{n=1}^{N} \frac{e^{-\lambda_n^2 \Delta t}}{N_{j,n}} \psi_{j,n}(\xi_j) \left(k_{xj} \frac{d\psi_{j,n}}{d\xi} \right)_{\xi=\xi_{j+1}} \int_0^{\Delta t} e^{\lambda_n^2 t'} t' \, dt' \quad , \qquad (19)$$

and the source term is expressed by

$$\begin{aligned} P_j^{m-1} = & \left(\frac{1}{\Delta t} \sum_{n=1}^{N} \frac{e^{-\lambda_n^2 \Delta t}}{N_{j,n}} \psi_{j,n}(\xi_j) \left(k_{xj} \frac{d\psi_{j,n}}{d\xi} \right)_{\xi=\xi_{j-1}} \int_0^{\Delta t} e^{\lambda_n^2 t'} (\Delta t - t') \, dt' \right) T_{j-1}^{m-1} \\ & - \left(\frac{1}{\Delta t} \sum_{n=1}^{N} \frac{e^{-\lambda_n^2 \Delta t}}{N_{j,n}} \psi_{j,n}(\xi_j) \left(k_{xj} \frac{d\psi_{j,n}}{d\xi} \right)_{\xi=\xi_{j+1}} \int_0^{\Delta t} e^{\lambda_n^2 t'} (\Delta t - t') \, dt' \right) T_{j+1}^{m-1} \\ & + \sum_{n=1}^{N} \frac{e^{-\lambda_n^2 \Delta t}}{N_{j,n}} \psi_{j,n}(\xi_j) \overline{F}_{j,n}^{m-1} \quad , \end{aligned} \qquad (20)$$

with

$$\lambda_{j,n} = \sqrt{\alpha_i} \frac{n\pi}{2\delta\xi_j} \quad , \qquad (21)$$

$$\psi_{j,n}(\xi) = sin(\frac{n\pi}{\delta\xi_j}(\xi - \xi_{j-1})) \quad , \qquad (22)$$

$$N_{j,n} = \frac{\delta\xi_j}{2} \quad , \qquad (23)$$

and

$$\delta\xi_j = \xi_{j+1} - \xi_{j-1} \quad . \qquad (24)$$

Generally, $N = 20$ provides results accurate enough for most applications. However, a larger number of eigenvalues may be necessary for calculating small-time solutions. The $\overline{F}_{j,n}^{m-1}$ term can be evaluated by

$$\overline{F}_{j,n}^{m-1} = \int_{R^{(j)}} \frac{k_{xj}}{\alpha_{xj}} \psi_{j,n}(\xi) F_j^{m-1}(\xi) \, d\xi \quad , \qquad (25)$$

where the function $F_j^{m-1}(\xi)$ is approximated by second-order Lagrange polynomials passing through T_{j-1}^{m-1}, T_j^{m-1} and T_{j+1}^{m-1}, that is

$$F_j^{m-1}(\xi) = \Psi_{j-1}(\xi) T_{j-1}^{m-1} + \Psi_j(\xi) T_j^{m-1} + \Psi_{j+1}(\xi) T_{j+1}^{m-1} \quad , \qquad (26)$$

with

$$\Psi_{j-1}(\xi) = \frac{(\xi - \xi_j)(\xi - \xi_{j+1})}{(\xi_{j-1} - \xi_j)(\xi_{j-1} - \xi_{j+1})} \quad , \qquad (27)$$

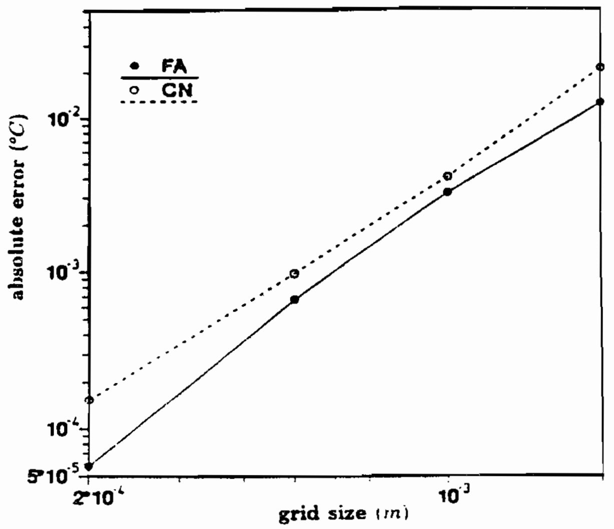

Figure 2: Absolute error ϵ_{abs} versus grid size $\Delta\xi$.

$$\Psi_j(\xi) = \frac{(\xi - \xi_{j-1})(\xi - \xi_{j+1})}{(\xi_j - \xi_{j-1})(\xi_j - \xi_{j+1})} , \qquad (28)$$

$$\Psi_{j+1}(\xi) = \frac{(\xi - \xi_{j-1})(\xi - \xi_j)}{(\xi_{j+1} - \xi_{j-1})(\xi_{j+1} - \xi_j)} . \qquad (29)$$

Similar formulas can be derived for each unknown nodal point j, k and l. In the present study, considering that the $R^{(j)}$ elements do not span accross the boundary between two different composite layers, the above formulation shall be completed converting into an algebraic form, with a finite-difference approximation, the interface conditions given by Eqs.(8) and (9). This system of algebraic equations, relating any node to its neighboring nodal values, can be solved to provide the FA numerical solution of the problem.

5. NUMERICAL RESULTS

In order to illustrate the effectiveness of the ADI FA numerical model developed previously, two sample problems were considered. The FA solutions were evaluated in comparison with known analytical solutions and with an ADI Cranck-Nicolson (CN) finite-difference model. All computations were carried out on a HP/Apollo 9000/425t workstation. A detailed study of the accuracy and the efficiency of the FA method for solving conduction problems in composite structures is reported by Ramos (1992).

The first sample problem consists of a homogeneous 2D plate ($\alpha = 5.4 \times 10^{-7}\ m^2/s$, $k = 1.1\ W/mK$) of side $L = 0.01\ m$, with initial unit uniform temperature and convective boundary condition with respect to an environment at $T_\infty = 0°C$. The heat transfer coefficient h is constant and uniform throughout the plate sides. To study the convergence of the FA solution, the absolute error ϵ_{abs}, defined as

$$\epsilon_{abs} = | T_{num} - T_{exact} | , \qquad (30)$$

was computed in a plate vertex, at $t = 5\,s$, and plotted against the grid size $\Delta\xi$, for $s = \alpha\Delta t/\Delta\xi^2 = 0.14$. The results, presented in Fig.2, show that the FA method has a second-order convergence rate, similar to that presented by the CN solution. The efficiency of the FA method was assessed by comparing the CPU time for both the FA and the CN methods to achieve the same level of accuracy. The results are plotted in Fig.3. As expected, for a prescribed accuracy level, the FA approach requires a much coarser grid and hence less CPU time than that required by the CN method. For example, at $\epsilon_{abs} = 1.5 \times 10^{-4} K$, the FA method is almost 10 times faster than the CN method.

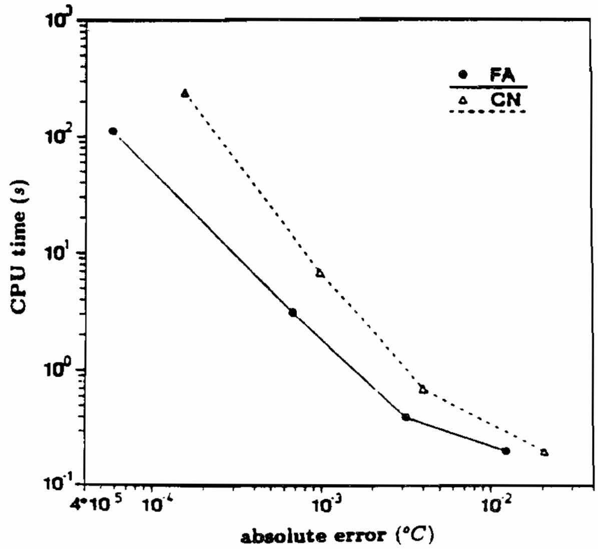

Figure 3: CPU time versus absolute error ϵ_{abs} ($s = 0.14$; 6×6, 11×11, 21×21 and 51×51 nodes).

The second example concerns a homogeneous 3D cube with initial zero uniform temperature and adiabatic boundary conditions ($h = 0$). For $t > 0$, the front face is submitted to a plane square-wave heat pulse of duration $\tau = 1$, 10 or 30 s and with a flux intensity adjusted to generate a steady-state temperature of $T_{ss} = 1$, 2 or $3°C$, respectively. The temperature history of the front face is shown in Fig.4. The FA results agree very well with the exact solution with an average relative error of less than 0.7 %.

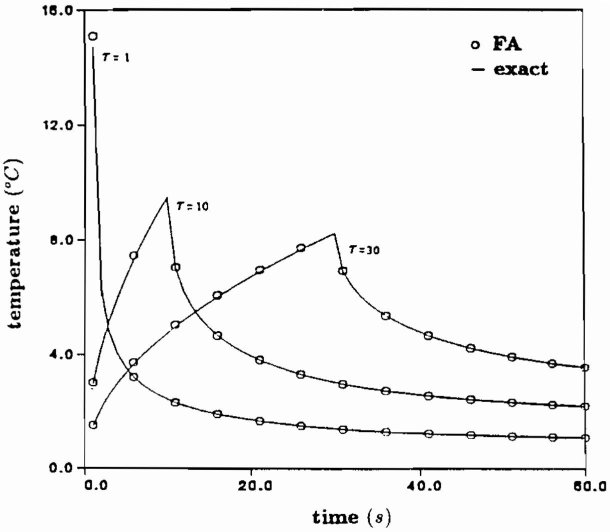

Figure 4: Temperature histories at the front face of a homogeneous cube, subjected to heat pulses of different intensities ($\Delta t = 1\ s$, $11 \times 11 \times 11$ nodes).

6. EXPERIMENTAL RESULTS

To demonstrate the applicability of the ADI FA model in solving a practical problem of current technological interest, the 3D transient temperature response of a carbone-epoxy laminate sample containing internal defects was simulated. The numerical results were compared with those obtained in laboratory, during a pulsed photothermal NDE experiment.

The test sample is a $200 \times 200 \times 5\ mm$ flat plate, made of 17 layers of carbone fibers inbedded in an epoxy matrix. Carbone-epoxy laminates are low thermal conductivity, highly anisotropic materials ($\alpha_y = \alpha_z = 10 \times \alpha_x = 2.2 \times 10^{-6}\ m^2/s$), widely used in the aerospace industry. During the sample fabrication process, rectangular 180 μm thick teflon inclusions of various sizes have been inserted at different depths and locations, as shown in Fig.5, to simulate subsurface damages.

The test apparatus consists of a pulsed infrared (IR) source, an AGEMA 880 long-wave IR camera, placed perpendicularly to the sample front face, and a data acquisition system, allowing real-time 12-bits digitalisation and hard disk storage in a rate of 25 images per second. For $t > 0$, the front face is irradiated with a square-wave heat pulse, while a sequence of 504 thermographic images is recorded, covering both the heating and cooling phases.

Knowing the emissivity of the sample, time-varying thermograms can be extracted from the data by col-

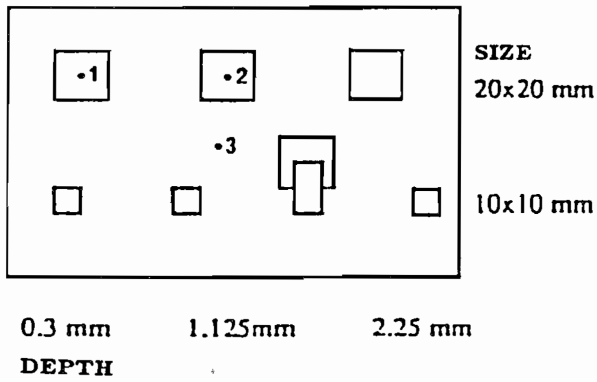

Figure 5: Defects arrangement in the carbone-epoxy sample.

lecting the thermal signal at particular *pixels* in the thermographic images. Experimental temperature histories for three different locations (points 1, 2 and 3 in Fig.5) are plotted in Fig.6 and 7 and show an excellent agreement with the results obtained by the ADI FA method.

7. CONCLUSION

In the present work, an alternating direction implicit (ADI) finite analytic (FA) numerical scheme is presented and applied to the solution of a transient heat conduction problem in a 3D multilayered composite plate. The basic idea of the FA method is the incorporation of a local analytic solution of the governing equation in the numerical solution of the boundary-value problem. The FA method allows for a coarser computational grid than the standard numerical schemes and hence provides savings in processing time. Numerical and experimental results show that the FA method is accurate and well suited for simulating the transient temperature response of composite samples during a pulsed photothermal NDE experiment.

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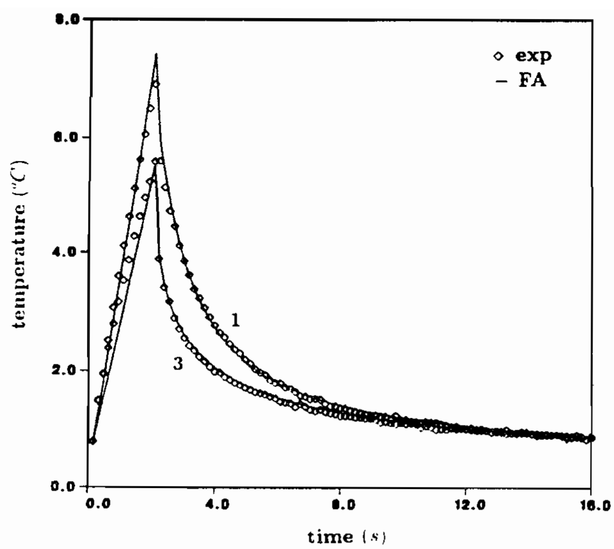

Figure 6: Experimental and numerical thermograms at points 1 and 3 of the carbone-epoxy sample.

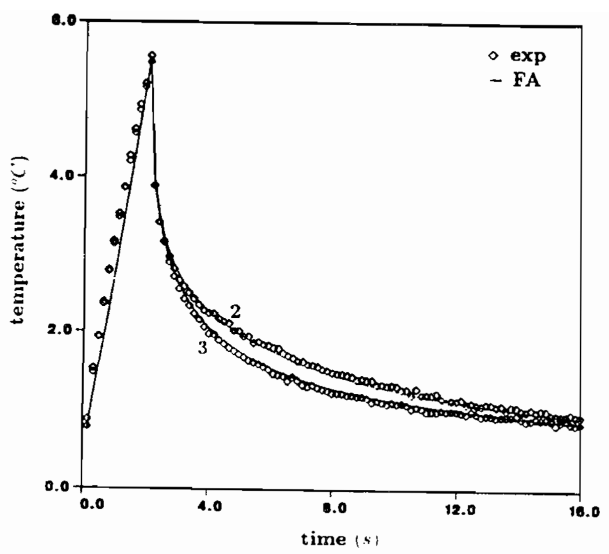

Figure 7: Experimental and numerical thermograms at points 2 and 3 of the carbone-epoxy sample.

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