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AN OPTIMAL LINEAR ESTIMATION AND PROJECTION OF THE GRADIENT APPROACH TO SOLVE LINEAR PROGRAMMING PROBLEMS

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ABSTRACT

A method to solve linear programming problems is presented. It combines an optimal linear estimation approach to solve systems of linear algebraic equations with the projection of the gradient method. This results in a solution search procedure wich can involve interior and or boundary points and which is expected to have satisfactory numerical performance and complexity.

1. INTRODUCTION

Schemes where parameterized suboptimal solutions are used in order to have a more realistic modelling of problems are frequently adopted in applied optimal control. This reduces the numerical solution of dynamic control problems to one of parameter optimization in each iteration. Linear programming or projection of the gradient type methods are then good mathematical programming tools to be used in these schemes (Ceballos and Rios Neto, 1981; Prado and Rios Neto, 1990), leading to insights that conduct one to arrive at approaches like that proposed here to solve the usual linear programming problem (P).

The combination of an approach to solve linear systems (Freitas Pinto and Rios Neto, 1990; Abbaffy and Spedicato, 1984) with the projection of the gradient method results in a method to solve the linear programming problem (P) with favorable characteristics concerning numerical complexity and performance, expected to be competitive with path-following type methods (Gonzaga, 1992).

Good numerical complexity is expected to be attained as a consequence of the resulting method algorithm conducting the search in each step looking for a possible direction closest to the opposite of the objective function gradient. This results in a solution search procedure which can involve interior and or boundary points, with the search along this last type of points not being restricted to go from vertex to vertex as in the Simplex method.

Favorable characteristics in terms of numerical performance, specially when dealing with problems of round-off errors and ill-conditioned linear systems, are expected as a consequence of features that allow both iterative schemes and factorized forms to be used in the approach to solve linear systems (Freitas Pinto and Rios Neto, 1990).

To present the proposed method a heuristic approach is used and the paper is organized as follows in the next sections. The method basic procedure and algorithm are presented in Section 2. Section 3 presents a few remarks form a preliminary analysis. Section 4 presents the paper conclusions.

2. METHOD BASIC PROCEDURE

The objective is to solve the standard linear programming problem (P):

$$\text{Minimize: } c^T x; \quad \text{subject to} \quad Ax = b,\ x \geq 0 \qquad (2.1)$$

where x is a real n dimensional vector; A is a real mxn matrix of rank m, formed of row vectors $a_1, a_2, \ldots, a_m$; all other vectors are real of appropriate dimensions; and the problem bounded with an optimal basic feasible solution (see, for example, Luemberger, 1973).

To do so, one combines the projection of gradient method with the approach by Freitas Pinto and Rios Neto (1990) to solve the constrained optimization problem:

$$\text{Minimize: } \frac{1}{2}(x - \bar{x})^T (x - \bar{x}); \quad \text{subject to} \quad A^e x = b^e \qquad (2.2)$$

with the following algorithm:

(i) Take $x^0 = \bar{x}$, $P_0 = I_n$

(ii) For $i=1,2,\ldots,m$ compute

$$x^i = x^{i-1} + \left(b_i^e - a_i^e x^{i-1}\right) p_i; \quad p_i = q_i P_{i-1} \left(a_i^e\right)^T \qquad (2.3)$$

$$q_i = \left(a_i^e P_{i-1} \left(a_i^e\right)^T\right)^{-1}; \qquad P_i = P_{i-1} - p_i a_i^e P_{i-1} \quad (2.4)$$

getting $x^e = x^m$ as the point in the hyperplane of (2.2) closest to $\bar{x}$ and $P^e = P_m$ the projection matrix associate to A^e.

The result is the following algorithm for searching a solution to problem (**P**) of (2.1):

Step 1: In correspondence with a feasible point $\mathbf{x}_f$, identify the active constraints in (**P**) and redefine A as an extended A^e to include these active constraints.

Step 2: Using the algorithm of (2.3) and (2.4), recursively calculate the projection matrix P^e associated to A^e and determine the projection of the opposite of the objective function gradient:

$$d = - P^e c \tag{2.5}$$

Step 3: If $d \neq 0$, recalculate x_f as:

$$x_f \leftarrow x_f + f\, d \tag{2.6}$$

choosing the maximum value of factor f such as to still have a feasible point; return to Step 1.

Step 4: If $d = 0$, calculate the inverse $[(A^e)^T]^{-1}$ of $(A^e)^T$, using the algorithm of (2.3), (2.4). Determine:

$$g = -\left[\left(A^e\right)^T\right]^{-1} c \tag{2.7}$$

and (i) if $g_j \le 0$ for all components in correspondence with active satisfied inequalities in (**P**) stop, Kuhn-Tucker conditions are satisfied;
(ii) if some of the $g_j > 0$, analyse as shown bellow which positivity constraints can be deactivated, redefine A^e and return to Step 2.

To decide in Step 4 (ii) which constraints can be deactivated, take the partition:

$$g^T = \left[g_p^T \,\vdots\, g_a^T\right], \qquad \left(A^e\right)^T = \left[\left(A_p^e\right)^T \,\vdots\, \left(A_a^e\right)^T\right] \tag{2.8}$$

where p is to indicate the $g_j > 0$ and a refers to active constraints. Calculate the projection matrix P_a^e and

$$d_a = - P_a^e c = P_a^e \left(A_a^e\right)^T g_p \neq 0 \tag{2.9}$$

and verify for which rows of A_p^e it is not true that

$$\left(a_p^e\right)_i d_a > 0 \tag{2.10}$$

redefining g_a and A_a^e to include in A_a^e the row for which the most negative value occurred in the verification in (2.10). Recalculate d_a in (2.9) verifying again (2.10) and reiterating the procedure until (2.10) is verified for all rows of the remaining A_p^e; noticing that in the worst case there will at least be one row in the remaining A_p^e and the resulting d_a will be a valid direction of search (Luemberger, 1973).

3. METHOD ANALYSIS

(i)Notice that for generating a feasible solution x_f in Step 1, one can always use the same method, solving in a first phase a problem with artificial variables as suggested by Luemberger (1973):

$$\text{Minimize: } \sum_{i=1}^{m} y_i; \quad \text{subject to } Ax + y = b,\ x \geq 0,\ y \geq 0 \quad (3.1)$$

assuming without loss of generality $b \geq 0$ and starting with $x=0$, $y=b$.

(ii) The algorithm of Section 2 leads to a kind of method where the search can result in a combination of going along interior and constraints boundary points. When the search is along the boundary it is not restricted to go from vertex to vertex as in the Simplex Method. Thus, though a rigorous analysis was not done, the method is expected to have polynomial complexity, since in each search step it has as intrinsic characteristic looking for a possible direction closest to the opposite of the objective function gradient.

(iii) The recursive nature of the procedure of (2.2) to (2.4) in Section 2 guarantees efficiency in the calculations of projection and pseudo inverses matrices needed along the use of the search algorithm. Besides that, to have better performance in ill conditioned problems one can always use the algorithm of (2.3) and (2.4) in a factorized from and or in an iterative scheme as proposed by Freitas Pinto and Rios Neto (1990).

4. CONCLUSIONS

Exploring ideas and results of a previous work (Freitas Pinto and Rios Neto, 1990), a method was proposed to solve the usual linear programming problem (P). This was done using an optimal linear parameter estimation type of method combined with the projection of the gradient method. The result was a method where a mixed kind of search can occur involving both interior and boundary points. A qualitative analysis of the method raises the expectation of polynomial complexity, though this should be rigorously treated before a definite conclusion can be made.

Previous experience with the linear estimation algorithm associated to the method indicates a good numerical performance (Freitas Pinto and Rios Neto, 1990), in terms of attenuating deterioration due to computer round-off.

Further studies should explore the analysis and determination of the characteristic of the method concerning computational complexity.

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