INDUCTANCE OF TOKAMAK PLASMAS USING TOROIDAL MULTIPOLAR EXPANSIONS

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1. Introduction

In a recent paper [1] it was shown how the external and mutual inductance coefficients of tokamak plasma equilibria can be determined in conformity with both a given plasma shape and external fields. The required link between the plasma current and the external fields was derived from the constraint imposed on the equivalent surface current by the total poloidal magnetic field distribution over the plasma boundary. This formulation clarified and extended the results of the work by Hirshman and Neilson [2], restricting the space of solutions consistent with the tokamak configuration. In the present paper a decomposition in toroidal coordinates is introduced, so that the boundary conditions can be satisfied by matching moments in the multipolar expansions for the poloidal flux due to the external currents. The matching expansions inside the plasma are calculated either from the equivalent surface current or in terms of the actual external sources. This procedure leads basically to the same results of Reference [1], in which the boundary conditions were satisfied by cancellation of harmonics in a Fourier series expansion for the poloidal flux at the plasma edge. The values of the external inductance obtained with the present method show good agreement with the previous results for a wide range of tokamak plasma aspect ratios.

2. Poloidal flux function in toroidal coordinates

In an axisymmetric toroidal plasma configuration the flux Φ between the symmetry axis and any flux surface is given in terms of the surface integral over the total toroidal current density j_T times the Green's function G for a toroidal ring current

$$\Phi(\overrightarrow{r}) = -\mu_0 \int\int j_T(\overrightarrow{r_1}) G(\overrightarrow{r}, \overrightarrow{r_1})\, d^2 r_1(\phi), \tag{1}$$

where $d^2 r_1(\phi)$ is the differential area element in the surface ϕ = constant of a cylindrical coordinate system (R, ϕ, Z). The subscript 1 indicates the coordinates of the sources.

In toroidal coordinates (η, ω, ϕ) the Green's function for a system with equatorial symmetry admits a decomposition of the form

$$G(\eta, \omega; \eta_1, \omega_1) = R_m \sum_{n=0}^{\infty} \left(\frac{2 - \delta_{n,0}}{n^2 - 1/4} \right) \frac{\cos(n\omega)\cos(n\omega_1) f_n(\eta_<) g_n(\eta_>)}{\sqrt{\cosh(\eta) - \cos(\omega)}\sqrt{\cosh(\eta_1) - \cos(\omega_1)}}, \tag{2}$$

where $f_n(\eta)$ and $g_n(\eta)$ are the toroidal functions (Fock functions) and the parameter of the toroidal coordinate system, $0 < R_m < \infty$, defines the position of the magnetic axis.

The poloidal flux in a tokamak is given by the sum of the flux Φ_{int} due to the internal sources, i.e., the plasma current density, and the flux Φ_{ext} due to the external sources. In general, the flux due to the external sources is given by the sum of the fluxes produced by an ideal transformer, Φ_M, by an uniform vertical equilibrium field, B_{vert}, and by a set of ring currents with equatorial symmetry, I_k, which represent the poloidal field coils

$$\Phi_{ext}(\overrightarrow{r}) = \Phi_M + \pi R^2 B_{vert} - \mu_0 \sum_k I_k G(\overrightarrow{r}, \overrightarrow{r_k}). \tag{3}$$

Assuming that all the poloidal field coils are outside the toroidal cavity η, the external flux admits the multipolar expansion [3]

$$\Phi_{ext}(\eta, \omega) = \frac{\mu_0 a I_T(a)}{\sqrt{\cosh(\eta) - \cos(\omega)}} \sum_{n=0}^{\infty} M_n^e(a) g_n(\eta) \cos(n\omega), \tag{4}$$

where a denotes the minor radius of the plasma, $I_T(a)$ is the total toroidal plasma current and the external moments are given by

$$\begin{aligned} M_n^e(a) \quad = \quad & \frac{1}{\sqrt{2}\pi} \left(\frac{2 - \delta_{n,0}}{n^2 - 1/4} \right) \frac{\Phi_M}{\mu_0 a I_T(a)} - 2\sqrt{2}(2 - \delta_{n0}) \left(\frac{R_m}{a} \right)^2 \frac{a B_{vert}}{\mu_0 I_T(a)} - \\ & \sum_k \left(\frac{2 - \delta_{n,0}}{n^2 - 1/4} \right) \left(\frac{R_m}{a} \right) \frac{I_k}{I_T(a)} \frac{f_n(\eta_k) \cos(n\omega_k)}{\sqrt{\cosh(\eta_k) - \cos(\omega_k)}}. \end{aligned} \tag{5}$$

Now, the poloidal flux $\Phi(0)$ between the magnetic axis and infinity is the sum of the fluxes Φ between the symmetry axis and a given flux surface and Φ_P between the magnetic axis and the same flux surface

$$\Phi(0) = \Phi + \Phi_P. \tag{6}$$

The initial conditions of the magnetizing circuit define the constant value $\Phi(0)$; it is zero for an initially unloaded transformer. Hence, the constant flux requirement at the plasma boundary provides a link between the internal and the external sources

$$\Phi(a) = \Phi_{int}(a) + \Phi_{ext}(a) = \Phi(0) - \Phi_P(a). \tag{7}$$

3. Equivalent surface current density

The vector analogue of Green's theorem gives an expression for the flux in the vacuum region, due to the plasma current, in terms of an equivalent surface current density

$$\Phi_{int}(\overrightarrow{r}) = -\mu_0 \oint K_T(\overrightarrow{r_1}) G(\overrightarrow{r}, \overrightarrow{r_1}) \, d\ell(\theta_1). \tag{8}$$

Here $d\ell(\theta)$ denotes the differential arc length along the poloidal angle θ in flux coordinates $(\rho, \theta, \zeta = -\phi)$. The toroidal component of the surface current density is given in terms of the gradient of the poloidal flux just inside the plasma-vacuum interface $\rho = a$; using the integral form of Ampère's law the surface current $K_T(a, \theta)$ is finally related by geometrical factors to the total toroidal current $I_T(a)$ [1]

$$K_T(a, \theta) = \left(\frac{\widehat{n} \cdot \nabla \Phi_P}{2\pi \mu_0 R} \right)_a = \frac{\left(|\nabla \rho| / R \right)_a I_T(a)}{\oint \left(|\nabla \rho| / R \right)_a d\ell(\theta)}. \tag{9}$$

Inside the plasma the equivalent surface current K_T, taken with the opposite sign, produces a magnetic field that coincides with the field produced by the external sources. This is a consequence of the vector analogue of Green's theorem and is equivalent to the principle of virtual casing. Indeed, the flux produced by the external sources inside the plasma can be written as

$$\Phi_{ext}(\overrightarrow{r}) = \Phi(0) - \Phi_P(a) + \mu_0 \oint K_T(\overrightarrow{r_1}) G(\overrightarrow{r}, \overrightarrow{r_1})\, d\ell(\theta_1), \tag{10}$$

so that, at the plasma boundary, the sum of Equations (10) and (8) satisfies exactly the constant flux requirement (7). Substituting Equations (9) and (2) in the integral (10), it follows that the external moments are also given by

$$\begin{aligned} M_n^e(a) &= \frac{1}{\sqrt{2}\pi}\left(\frac{2-\delta_{n,0}}{n^2-1/4}\right)\frac{\Phi(0)-\Phi_P(a)}{\mu_0 a I_T(a)} + \\ &\left(\frac{2-\delta_{n,0}}{n^2-1/4}\right)\frac{R_m/a}{\oint\left(|\nabla\rho|/R\right)d\ell(\theta)}\oint\left(\frac{|\nabla\rho|}{R}\right)_a \frac{f_n(\eta_a)\cos(n\omega_a)}{\sqrt{\cosh(\eta_a)-\cos(\omega_a)}}d\ell(\theta). \end{aligned} \tag{11}$$

4. External and mutual inductance coefficients

The expansion (4) can be used to represent the flux due to the external sources inside a toroidal cavity η contained by the plasma. An ideal representation of this flux is obtained taking the external moments calculated in terms of the equivalent surface current density by means of Equation (11). These same moments are calculated in terms of the actual sources by means of Equation (5). From the equivalence of the two expansions a set of matched toroidal moment equations ($n = 0, 1, 2 \ldots$) is obtained, relating the geometric parameters at the plasma-vacuum interface to the external sources.

The matched toroidal moment equations provide a very elegant and general way to link the internal and external sources in the tokamak equilibrium problem, by means of a single line integral for each toroidal moment. As mentioned in the Introduction, the present paper concerns the determination of the external inductance which is consistent with a given plasma shape and given external sources. Therefore, one takes the poloidal angle average of Equation (7) to obtain the flux balance condition

$$\Phi(0) - \Phi_P(a) = \langle \Phi_{int}(a)\rangle_\theta + \Phi_M + \pi \left\langle R^2(a)\right\rangle_\theta B_{vert} - \mu_0 \sum_k I_k \left\langle G[\overrightarrow{r}(a), \overrightarrow{r_k}]\right\rangle_\theta. \tag{12}$$

Defining the external inductance L_e, the mutual inductance coefficient between the plasma and the external magnetic field M, and the mutual inductance coefficients between the plasma and the external poloidal field coils M_k,

$$L_e = \frac{\langle\Phi_{int}(a)\rangle_\theta}{I_T(a)}, \qquad M = \frac{\langle R^2(a)\rangle_\theta}{R_0^2(a)}, \qquad M_k = -\mu_0\left\langle G[\overrightarrow{r}(a), \overrightarrow{r_k}]\right\rangle_\theta, \tag{13}$$

where $R_0(a)$ is the geometric center of the plasma cross-section, Equation (12) becomes

$$\Phi(0) - \Phi_P(a) = L_e I_T(a) + \Phi_M + \pi M R_0^2(a) B_{vert} + \sum_k M_k I_k. \tag{14}$$

This equation is used to calculate the external inductance once the other equilibrium quantities have been determined from a solution of the set of matched toroidal moment equations.

Equilibrium calculations for plasmas immersed in an uniform vertical equilibrium field were performed in order to compare the present results with the previously published results [1]. The plasma shape for equilibrium is given by a spectral representation specified by the values of the aspect-ratio, A, the geometrical elongation, $\kappa(a)$, and the geometrical triangularity, $\delta(a)$. This representation allows analytic evaluation of all the flux-surface averaged equilibrium coefficients [4]. The mutual inductance coefficient M, in particular, has a simple analytic expression for the assumed spectral representation of the plasma boundary [1]. The normalized value of the external vertical field, $aB_{vert}/[\ \mu_0 I_T(a)]$, specifies the external sources. Then, a set of matched toroidal moment equations is solved to determine the constant value of the normalized flux, $[\Phi_P(a) + \Phi_M - \Phi(0)]/[\mu_0 a I_T(a)]$, and the values of the free geometrical parameters (radial derivatives of A, κ and δ) at the plasma boundary.

Table 1 shows some of the results of the present calculations of the normalized inductance coefficients for several values of the aspect ratio. The input parameters $\kappa(a)$, $\delta(a)$ and $aB_{vert}/[\ \mu_0 I_T(a)]$ were taken from the previous results [1] for naturally elongated plasmas near the condition of maximum elongation. There is very good agreement between the previous calculation of the external inductance and the present one, with a maximum discrepancy of the order of $\sim 1.4\%$ in the extreme case $A = 1.1$. For aspect ratios larger than $A = 1.1$ the eventual difference in the last digit for the values of the external inductance in Table 1 is due to round-off approximations in the two calculations.

A	$\kappa(a)$	$\delta(a)$	$\frac{aB_{vert}}{\mu_0 I_T(a)}$	$\frac{L_e}{\mu_0 a}$	M
1.1	6.13	0.587	-0.0106	0.0497	1.145
1.15	3.72	0.570	-0.0264	0.135	1.128
1.2	2.85	0.541	-0.0419	0.234	1.120
1.3	2.08	0.475	-0.0666	0.444	1.112
1.4	1.74	0.411	-0.0817	0.646	1.108
1.5	1.55	0.355	-0.0900	0.837	1.104
1.7	1.36	0.266	-0.0959	1.197	1.095
2.0	1.23	0.178	-0.0944	1.734	1.080
3.0	1.10	0.0655	-0.0781	3.715	1.045
6.0	1.03	0.0116	-0.0506	11.36	1.013

Table 1. Results of the normalized inductance calculations for naturally elongated plasmas.

Acknowledgment. This work was supported by FAPESP - Fundação de Amparo à Pesquisa do Estado de São Paulo.

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