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Extended gravitoelectromagnetism. II. Metric perturbation

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The perturbation in the metric tensor is obtained in terms of the flat-space total energy-momentum tensor given by the sum of the consistent fully-relativistic fluid and gravitoelectromagnetic (GEM) field contributions. Expressions for the perturbed metric in the internal (fluid) and external (vacuum) regions are given in terms of the fluid and field variables. This formulation of gravitoelectromagnetism is compatible with the formation of gravitational waves. The geodesic equation is obtained both in the internal and external regions, including new terms neglected in the standard gravitomagnetic formulation.

I. INTRODUCTION

This paper picks up the extended gravitoelectromagnetic theory developed in the part I article (cf. the joint paper "Extended gravitoelectromagnetism. I. Variational formulation") [1]. A consistent set of equations of motion for a fully-relativistic perfect fluid immersed in a gravitoelectromagnetic field was obtained, in part I, by means of a variational formulation . The variational equations fully describe the dynamics of matter in the flat-space gravitational field. This approach to the problem led to an extended version of the original gravitomagnetic theory developed by Thirring [2–4].

In the present part II article the energy-momentum tensor derived in part I is used to obtain the space-time curvature effects according to the weak field approximation of Einstein's gravitational field equation. The gravitoelectromagnetic terms, forming both the interaction and the free-field Lagrangian in the flat-space variational formulation, correspond to a correction to first order, in the equation of motion, as measured by the strength G of the gravitational field. In this sense, the extended gravitoelectromagnetic formulation already gives a consistent first order post-Minkowski approximation. The total energy-momentum tensor formed by the sum of the fluid and gravitoelectromagnetic tensors satisfies the energy-momentum conservation equation to first order. This flat-space energy-momentum tensor forms a convenient source for the linearized Einstein equation. One may say that the equivalence between matter and gravitational energy is reinstated in the theory, allowing matter to exchange energy and momentum with the gravitational field. There is no conflict with Einstein's theory since the gravitoelectromagnetic field does not generate curvature, which is restricted to the mass portion of the tensor.

The paper is structured as follows. Section II reviews the equations which describe the dynamics of a fully relativistic perfect fluid in the gravitoelectromagnetic context. After a brief review of the linearized gravitational theory, in Section III, the total energy-momentum tensor formed by the sum of the consistent fluid and field tensors is taken as a source in the linearized theory of general relativity. The corrections to the metric components inside the fluid (Subsection III A) and in the vacuum outside the fluid (Subsection III B) can be obtained in a straightforward manner using the linearized gravitational field equations. The geodesic equations both inside the fluid source and in vacuum are derived in Section IV. Application of the geodesic equation is deferred to the final article in this three parts series. Section V gives the final comments and conclusions.

The most important content of the paper is in Section III, which describes the consistent weak gravitational field formulation of extended gravitoelectromagnetism.

II. FLUID DYNAMICS IN THE GRAVITOELECTROMAGNETIC FIELD

The variational procedure used in the part I article demonstrates that the equation for the evolution of the fluid velocity $\boldsymbol{u}$,

$$\frac{d}{dt}\left[\gamma\left(1+\frac{\gamma_A}{\gamma_A-1}\frac{\gamma k_B T}{mc^2}\right)\boldsymbol{u}\right]=-\frac{\boldsymbol{\nabla} p}{\rho}+\boldsymbol{E}_g+\boldsymbol{u}\times\boldsymbol{B}_g, \quad (1)$$

describes the relativistic fluid flow under the action of a self-consistent gravitoelectromagnetic (GEM) field in flat space. This equation shows the fully relativistic thermal motion effect on the fluid inertia, which increases the magnetic reconnection rate in astrophysical plasmas [5] but also affects neutral fluid flow. The equation of motion is complemented by the equation of continuity (conservation of the number of particles)

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{2}$$

and by Maxwell's source equations

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E}_g &= -4\pi G \rho && \text{GE (gravitoelectric) Gauss's law} \\ \boldsymbol{\nabla} \times \boldsymbol{B}_g &= -\frac{4\pi G}{c^2} \rho \boldsymbol{u} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}_g}{\partial t} && \text{GEM Ampère's law} \end{aligned} \tag{3}$$

The fluid pressure is denoted by p and the mass density ρ is given in terms of the number density n by $\rho = mn$. The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}, \tag{4}$$

and the "adiabatic" coefficient γ_A is a function of the fluid temperature [6]. The gravitational constant and the velocity of light are denoted by G and c, respectively. The gravitoelectromagnetic field variables $\boldsymbol{E}_g$ and $\boldsymbol{B}_g$ are related to the potentials ϕ_g and $\boldsymbol{A}_g$ by

$$\begin{aligned} \boldsymbol{E}_g &= -\boldsymbol{\nabla} \phi_g - \frac{\partial \boldsymbol{A}_g}{\partial t}, \\ \boldsymbol{B}_g &= \boldsymbol{\nabla} \times \boldsymbol{A}_g, \end{aligned} \tag{5}$$

which lead to the homogeneous Maxwell equations

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E}_g + \frac{\partial \boldsymbol{B}_g}{\partial t} &= 0 && \text{GEM Faraday's law} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B}_g &= 0 && \text{GM (gravitomagnetic) Gauss' law} \end{aligned} \tag{6}$$

The pressure p is related to the temperature T according to the isentropic flow condition $ds/dt = 0$ (conservation of energy). For a perfect fluid the equation of state $p = nk_BT$ relates the density n to the temperature T, closing the set of fluid equations. In their weakly relativistic form, these equations of fluid motion in a gravitoeletromagnetic field can be used to reproduce the observed shape of the galactic rotation curve without introducing dark matter [7].

Multiplying equation (1) by ρ and applying the continuity condition (2) gives the momentum density conservation equation

$$\begin{aligned} \frac{\partial}{\partial t} \left[\gamma \left(1 + \frac{\gamma_A}{\gamma_A - 1} \frac{\gamma p}{\rho c^2} \right) \rho \boldsymbol{u} \right] + \boldsymbol{\nabla} \cdot \left[\gamma \left(1 + \frac{\gamma_A}{\gamma_A - 1} \frac{\gamma p}{\rho c^2} \right) \rho \boldsymbol{u} \boldsymbol{u} + p \overline{\overline{\boldsymbol{I}}} \right] \\ = \rho \boldsymbol{E}_g + \boldsymbol{j} \times \boldsymbol{B}_g, \end{aligned} \tag{7}$$

where the equation of state $p = nk_BT$ was used to eliminate the temperature T. Here $\boldsymbol{j} = \rho \boldsymbol{u}$ is the mass current density and $\overline{\overline{\boldsymbol{I}}}$ denotes the unit dyadic. Scalar multiplication of this equation by $\boldsymbol{u}$ and use of the equations of continuity and state leads to the energy density conservation equation

$$\frac{\partial}{\partial t} \left[\gamma \rho c^2 + \left(\frac{1}{\gamma_A - 1} + \beta^2 \right) \gamma^2 p \right] + \boldsymbol{\nabla} \cdot \left[\left(\gamma \rho c^2 + \frac{\gamma_A}{\gamma_A - 1} \gamma^2 p \right) \boldsymbol{u} \right] = \boldsymbol{j} \cdot \boldsymbol{E}_g. \tag{8}$$

The contravariant form of the mass current density four-vector is given by

$$j^\mu = (\rho c, \boldsymbol{j}), \tag{9}$$

so that the covariant equation of continuity becomes

$$\partial_\mu j^\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \boldsymbol{\nabla} \right) \cdot (\rho c, \boldsymbol{j}) = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{10}$$

Defining the gravitoelectromagnetic field tensor by

$$F_g^{\mu\nu} = \begin{pmatrix} 0 & \boldsymbol{E}_g/c \\ -\boldsymbol{E}_g/c & \overline{\overline{\overline{\boldsymbol{\epsilon}}}} \cdot \boldsymbol{B}_g \end{pmatrix}, \tag{11}$$

where $\overline{\overline{\overline{\boldsymbol{\epsilon}}}}$ is the totally antisymmetric Levi-Civita tensor in three dimensions, the Maxwell source equations are written as

$$\partial_\nu F_g^{\mu\nu} = -\frac{4\pi G}{c^2} j^\mu. \tag{12}$$

In components form

$$\left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \cdot \begin{pmatrix} 0 & \boldsymbol{E}_g/c \\ -\boldsymbol{E}_g/c & \overline{\overline{\overline{\boldsymbol{\epsilon}}}} \cdot \boldsymbol{B}_g \end{pmatrix}^T = -\frac{4\pi G}{c^2} \left(\rho c, \boldsymbol{j}\right). \tag{13}$$

Using $\boldsymbol{\nabla}\cdot \left(\overline{\overline{\overline{\boldsymbol{\epsilon}}}} \cdot \boldsymbol{B}_g\right)^T = \boldsymbol{\nabla}\cdot \left(\overline{\overline{\overline{\boldsymbol{\epsilon}}}}^T \cdot \boldsymbol{B}_g\right) = -\boldsymbol{\nabla}\cdot \left(\overline{\overline{\overline{\boldsymbol{\epsilon}}}} \cdot \boldsymbol{B}_g\right) = -\boldsymbol{\nabla}\times\boldsymbol{B}_g$, where the superscript T denotes the transposed tensor, the above equation corresponds to the gravitoelectromagnetic laws of Gauss and Ampère (3).

The anti-symmetric field tensor is given in terms of the four-vector potential

$$A_g^\mu = \left(\phi_g/c, \boldsymbol{A}_g\right) \tag{14}$$

as follows

$$F_g^{\mu\nu} = \partial^\mu A_g^\nu - \partial^\nu A_g^\mu, \tag{15}$$

that is,

$$\begin{aligned} \begin{pmatrix} 0 & \boldsymbol{E}_g/c \\ -\boldsymbol{E}_g/c & \overline{\overline{\overline{\boldsymbol{\epsilon}}}} \cdot \boldsymbol{B}_g \end{pmatrix} &= \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \boldsymbol{\nabla} \end{pmatrix} \begin{pmatrix} \frac{\phi_g}{c} & \boldsymbol{A}_g \end{pmatrix} - \left[\begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \boldsymbol{\nabla} \end{pmatrix} \begin{pmatrix} \frac{\phi_g}{c} & \boldsymbol{A}_g \end{pmatrix}\right]^T \\ &= \begin{pmatrix} 0 & -\frac{1}{c}\frac{\partial \boldsymbol{A}_g}{\partial t} - \frac{1}{c}\boldsymbol{\nabla}\phi_g \\ \frac{1}{c}\boldsymbol{\nabla}\phi_g + \frac{1}{c}\frac{\partial \boldsymbol{A}_g}{\partial t} & \boldsymbol{\nabla}\boldsymbol{A}_g - \left(\boldsymbol{\nabla}\boldsymbol{A}_g\right)^T \end{pmatrix}, \end{aligned} \tag{16}$$

which corresponds to the relations (5) between the fields $\boldsymbol{E}_g$, $\boldsymbol{B}_g$ and the potentials ϕ_g, $\boldsymbol{A}_g$.

The quantities $A_g^\mu(x)$ and $A_g^\mu(x) - \partial^\mu f(x)$ are physically indistinguishable, so that A_g^μ can be required to satisfy Lorenz's condition

$$\partial_\mu A_g^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \cdot \left(\frac{\phi_g}{c}, \boldsymbol{A}_g\right) = \frac{1}{c^2}\frac{\partial \phi_g}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}_g = 0. \tag{17}$$

The inhomogeneous field equations can be written in terms of the four-potential as

$$\Box^2 A_g^\mu = \frac{4\pi G}{c^2} j^\mu \quad \begin{cases} \Box^2 \phi_g &= 4\pi G \rho \\ \Box^2 \boldsymbol{A}_g &= \frac{4\pi G}{c^2}\boldsymbol{j} \end{cases} \tag{18}$$

Taking into account the momentum (7) and energy (8) density conservation equations, the energy-momentum tensor of a perfect fluid can be defined by

$$T_f^{\mu\nu} = p\eta^{\mu\nu} + \left(\mathring{U} + p\right)\frac{u^\mu u^\nu}{c^2}, \tag{19}$$

where $\eta^{\mu\nu}$ is the Minkowski (flat-space metric) tensor

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\overline{\boldsymbol{I}}} \end{pmatrix} = \eta^{\mu\nu}, \tag{20}$$

$u^\mu = \gamma\left(c, \boldsymbol{u}\right)$ is the contravariant form of the fluid four-velocity, and $\mathring{U}$ is the proper energy density, sum of the rest mass and thermal energy densities,

$$\mathring{U} = \mathring{n}mc^2 + \frac{p}{\gamma_A - 1} = \frac{\rho c^2}{\gamma} + \frac{p}{\gamma_A - 1}. \tag{21}$$

The number density $\overset{\circ}{n}$ in the rest frame, denoted by the upper circle label, is given in terms of the number density n in the moving frame by $\overset{\circ}{n} = n/\gamma$. Accordingly, the temporal and spatial components of

$$\partial_\nu T_f^{\mu\nu} = j_\nu F_g^{\mu\nu} \tag{22}$$

correspond to the energy and momentum density conservation equations, respectively. Indeed,

$$\begin{aligned} \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \cdot \begin{pmatrix} \gamma^2 \left(\overset{\circ}{U} + p\beta^2\right) & \left(\overset{\circ}{U} + p\right)\gamma^2 \boldsymbol{u}/c \\ \left(\overset{\circ}{U} + p\right)\gamma^2 \boldsymbol{u}/c & p\overline{\overline{\boldsymbol{I}}} + \left(\overset{\circ}{U} + p\right)\gamma^2 \boldsymbol{u}\boldsymbol{u}/c^2 \end{pmatrix}^T \\ = (-\rho c, \boldsymbol{j}) \cdot \begin{pmatrix} 0 & \boldsymbol{E}_g/c \\ -\boldsymbol{E}_g/c & \overline{\overline{\overline{\boldsymbol{\epsilon}}}} \cdot \boldsymbol{B}_g \end{pmatrix}^T \end{aligned} \tag{23}$$

The energy conservation equation is given by the temporal component of the above equation

$$\frac{1}{c}\frac{\partial}{\partial t}\left[\gamma^2\left(\overset{\circ}{U} + p\beta^2\right)\right] + \boldsymbol{\nabla} \cdot \left[\left(\overset{\circ}{U} + p\right)\gamma^2 \frac{\boldsymbol{u}}{c}\right] = \boldsymbol{j} \cdot \frac{\boldsymbol{E}_g}{c}, \tag{24}$$

and the momentum conservation equation is given by the spatial components

$$\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\overset{\circ}{U} + p\right)\gamma^2 \frac{\boldsymbol{u}}{c}\right] + \boldsymbol{\nabla} \cdot \left[\left(\overset{\circ}{U} + p\right)\gamma^2 \frac{\boldsymbol{u}\boldsymbol{u}}{c^2} + p\overline{\overline{\boldsymbol{I}}}\right] = \rho \boldsymbol{E}_g + \boldsymbol{j} \times \boldsymbol{B}_g. \tag{25}$$

Note that $-\boldsymbol{j} \cdot \left(\overline{\overline{\overline{\boldsymbol{\epsilon}}}} \cdot \boldsymbol{B}_g\right) = \boldsymbol{j} \times \boldsymbol{B}_g$. With some rearrangement these equations can be written in the form of the momentum density (7) and energy density (8) conservation equations.

The thermodynamic potentials in the rest frame satisfy the second law

$$\overset{\circ}{T} ds = d\left(\overset{\circ}{U}/\overset{\circ}{n}\right) + pd\left(1/\overset{\circ}{n}\right), \tag{26}$$

where s is the specific entropy and $\overset{\circ}{T} = \gamma T$ is the rest frame temperature.

In summary, the dynamics of a fully relativistic perfect fluid in the flat-space gravitoelectromagnetic field is governed by the covariant set of equations

$$\begin{cases} \partial_\nu T_f^{\mu\nu} = j_\nu F_g^{\mu\nu} & \text{energy-momentum conservation} \\ \Box^2 A_g^\mu = \dfrac{4\pi G}{c^2} j^\mu & \text{Maxwell's source equations} \\ p u^\mu \partial_\mu s = 0 & \text{entropy conservation} \end{cases} \tag{27}$$

with

$$F_g^{\mu\nu} = \partial^\mu A_g^\nu - \partial^\nu A_g^\mu \;\text{ and }\; \partial_\mu A_g^\mu = 0. \tag{28}$$

This set of covariant equations gives a total of 9 equations in the 5 fluid $(\rho, \boldsymbol{u}, p)$ and 4 field $(\phi_g, \boldsymbol{A}_g)$ variables. Note that the continuity condition is automatically satisfied by Maxwell's source equations constrained by Lorenz's condition for A_g^μ. Note also that the energy-momentum equation constrained by the second law of thermodynamics satisfies the condition of isentropic flow [1]. Taking into account the second law (26) only one equation of state, the perfect gas law in the present case, is needed to close the system of fluid-field equations. In general, the temperature can be considered as defined by the second law.

The fluid energy-momentum tensor $T_f^{\mu\nu}$ can be written in components form as follows

$$T_f^{\mu\nu} = \begin{pmatrix} U_f & c\boldsymbol{G}_f \\ c\boldsymbol{G}_f & \overline{\overline{\boldsymbol{T}}}_f \end{pmatrix}, \tag{29}$$

where the energy, momentum and stress densities for the fluid are defined by

$$\begin{aligned} U_f &= \gamma\rho c^2 + \left(\frac{1}{\gamma_A - 1} + \beta^2\right)\gamma^2 p && \text{energy density} \\ \boldsymbol{G}_f &= \left(\gamma\rho + \frac{\gamma_A}{\gamma_A - 1}\frac{\gamma^2 p}{c^2}\right)\boldsymbol{u} && \text{momentum density} \\ \overline{\overline{\boldsymbol{T}}}_f &= \gamma\rho\boldsymbol{u}\boldsymbol{u} + \left(\overline{\overline{\boldsymbol{I}}} + \frac{\gamma_A}{\gamma_A - 1}\gamma^2\frac{\boldsymbol{u}\boldsymbol{u}}{c^2}\right)p && \text{stress density} \end{aligned} \tag{30}$$

Using Maxwell's source equations the four-current density $j^\mu = (\rho c, \boldsymbol{j})$ can be eliminated from the right-hand side of the energy-momentum conservation equation by defining the energy, momentum and stress densities for the field

$$\begin{aligned} U_g &= -\frac{E_g^2 + c^2 B_g^2}{8\pi G} = \overline{\overline{\boldsymbol{T}}}_g : \overline{\overline{\boldsymbol{I}}} && \text{energy density} \\ \boldsymbol{G}_g &= -\frac{1}{4\pi G}\left(\boldsymbol{E}_g \times \boldsymbol{B}_g\right) = \frac{\boldsymbol{S}_g}{c^2} && \text{momentum density} \\ \overline{\overline{\boldsymbol{T}}}_g &= -\frac{1}{4\pi G}\left(\frac{E_g^2}{2}\overline{\overline{\boldsymbol{I}}} - \boldsymbol{E}_g\boldsymbol{E}_g\right) - \frac{c^2}{4\pi G}\left(\frac{B_g^2}{2}\overline{\overline{\boldsymbol{I}}} - \boldsymbol{B}_g\boldsymbol{B}_g\right) && \text{stress density} \end{aligned} \tag{31}$$

so that the gravitational field energy-momentum tensor can be defined by

$$T_g^{\mu\nu} = \begin{pmatrix} U_g & c\boldsymbol{G}_g \\ c\boldsymbol{G}_g & \overline{\overline{\boldsymbol{T}}}_g \end{pmatrix}. \tag{32}$$

The vector of Poynting is denoted by $\boldsymbol{S}_g = c^2 \boldsymbol{G}_g$. The full energy-momentum equation which describes the flat space dynamics of a fluid in the gravitoelectromagnetic field can be written in covariant form as follows

$$\partial_\nu T^{\mu\nu} = 0. \tag{33}$$

where $T^{\mu\nu}$ is the total energy-momentum tensor given by the sum of the fluid $T_f^{\mu\nu}$ and gravitoelectromagnetic $T_g^{\mu\nu}$ tensors:

$$T^{\mu\nu} = T_f^{\mu\nu} + T_g^{\mu\nu} = \begin{pmatrix} U_f + U_g & c\left(\boldsymbol{G}_f + \boldsymbol{G}_g\right) \\ c\left(\boldsymbol{G}_f + \boldsymbol{G}_g\right) & \overline{\overline{\boldsymbol{T}}}_f + \overline{\overline{\boldsymbol{T}}}_g \end{pmatrix}. \tag{34}$$

The equation $\partial_\nu T^{\mu\nu} = 0$ describes the exchange of energy between matter and the gravitoelectromagnetic field in flat space. As shown in the part I article, the full set of variational fluid-field equations supports the formation of gravitoelectromagnetic waves, as originally deduced by Heaviside [8]. These waves are based on a vector field, reproducing many, but not all, of the general relativity features of the tensor-based gravitational waves [9]. Clearly, the gravitoelectromagnetic is a vector field, constituting only part of the full tensorial solution to gravitational theory. Nevertheless, the variational formulation gives a strong physical basis for the theory in flat space. The connection between the gravitoelectromagnetic and gravitational theories is explored in the next section.

III. WEAK-FIELD APPROACH TO GRAVITOELECTROMAGNETISM

In the linearized gravitational field theory the metric tensor $g_{\mu\nu}$ is treated as a linear perturbation from the flat-space metric $\eta_{\mu\nu}$ defined by equation (20):

$$g_{\mu\nu}\left(x^\mu\right) = \eta_{\mu\nu} + h_{\mu\nu}\left(x^\mu\right). \tag{35}$$

Since both $g_{\mu\nu}$ and $\eta_{\mu\nu}$ are symmetric, the metric perturbations $h_{\mu\nu}$ are symmetric functions of $x^\mu = (ct, \boldsymbol{r})$, and are assumed of small magnitude $|h_{\mu\nu}| \ll 1$. By neglecting terms of second and higher order in $|h_{\mu\nu}|$, the weak-field approximation to Einstein's field equation leads to the following set of equations [10–13],

$$\begin{cases} \Box^2 \widetilde{h}_{\mu\nu} = -\dfrac{16\pi G}{c^4} T_{\mu\nu} & \text{linearized Einstein equation} \\ \partial^\nu \widetilde{h}_{\mu\nu} = 0 & \text{harmonic gauge} \end{cases} \tag{36}$$

where $\widetilde{h}_{\mu\nu}$ is the trace-reversed metric perturbation

$$\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h. \tag{37}$$

The trace of $\widetilde{h}_{\mu\nu}$ is

$$\widetilde{h} = \eta^{\mu\nu}\widetilde{h}_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} - 2h = -h, \tag{38}$$

which is the trace of $h_{\mu\nu}$ with the sign reversed. The actual metric perturbation $h_{\mu\nu}$ is given in terms of the trace-reversed perturbation by

$$h_{\mu\nu} = \widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h}, \tag{39}$$

and the linearized metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h}. \tag{40}$$

The linearized field equation (36) implies that the energy-momentum tensor must satisfy the consistency condition (equivalent to the harmonic gauge)

$$\partial^\nu T_{\mu\nu} = 0. \tag{41}$$

In the weak-field approximation the indices of a quantity that is of order of magnitude of $h_{\mu\nu}$ can be raised or lowered using the Minkowski metrics $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively, instead of $g^{\mu\nu}$ or $g_{\mu\nu}$. Therefore, to order $|h_{\mu\nu}| \ll 1$, the harmonic gauge is equivalent to the flat space energy-momentum conservation equation in (33)

$$\partial_\nu T^{\mu\nu} = 0. \tag{42}$$

The question is how to extend the definition of $T^{\mu\nu}$ so that the exact Einstein's field equation is satisfied, to all orders of $|h_{\mu\nu}|$, while the modified energy-momentum for a gravitational field plus the contained matter is conserved. This is achieved, in a general form, by means of the pseudo-tensor reformulation of Einstein's equation advanced by Landau and Lifshitz [10]. In the present article it is proposed to define $T_{\mu\nu}$ by (34), which may lead to a solution that differs from the exact one by terms of second-Minkowski order G^2. Eventual spurious terms or indefiniteness introduced in the metric perturbations are of second Minkowski-order. They can be presumably corrected afterwards, by adjusting boundary conditions. An example is given in Section IV for the derivation of the geodesic equation in the vacuum region, where the far-field solution is adjusted to the correct energy limit.

Considering the formulation presented in Section II, the dynamics of a perfect fluid in the gravitational weak-field approximation is governed by

$$\begin{cases} \Box^2 \widetilde{h}_{\mu\nu} &= -\dfrac{16\pi G}{c^4} T_{\mu\nu} & \text{field equation} \\ \partial^\nu \widetilde{h}_{\mu\nu} &= 0 & \text{harmonic gauge} \\ p u^\mu \partial_\mu s &= 0 & \text{specific entropy conservation} \end{cases} \tag{43}$$

where $T_{\mu\nu}$ already constitutes a consistent first Minkowski-order solution as given by equations (33) and (34). The field equation (linearized Einstein equation) constitutes ten equations relating the ten components of the symmetric metric tensor perturbation $\widetilde{h}_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$. The harmonic gauge gives four more equations for $\widetilde{h}_{\mu\nu}$. Finally, the entropy conservation equation gives one equation relating the fluid variables (equation of state). Hence, there is a total of fifteen equations for the ten components $\widetilde{h}_{\mu\nu}$ of the metric and for the five fluid variables, namely, mass density ρ, pressure p and components of the fluid velocity $\boldsymbol{u}$. The full metric tensor is given by equation (40):

$$g_{\mu\nu} = \eta_{\mu\nu} + \widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h}. \tag{44}$$

It remains to relate the fluid variables to the scalar ϕ_g and vector $\boldsymbol{A}_g$ potentials of the gravitoelectromagnetic field. These four relations are provided by the energy-momentum conservation equation in flat space (33), which is required for consistency of both the field equation and the harmonic gauge, at least to first order in $|h_{\mu\nu}|$:

$$\partial_\nu T^{\mu\nu} = 0. \tag{45}$$

According to the variational principle developed in the part I article, this equation describes the dynamics of a relativistic fluid immersed in the gravitoelectromagnetic field in flat space. As reviewed in Section II, the total energy-momentum tensor $T^{\mu\nu}$ is given by the sum (34) of the fluid $T_f^{\mu\nu}$ and gravitoelectromagnetic $T_g^{\mu\nu}$ tensors:

$$T^{\mu\nu} = T_f^{\mu\nu} + T_g^{\mu\nu} = \begin{pmatrix} U_f + U_g & c\,(\boldsymbol{G}_f + \boldsymbol{G}_g) \\ c\,(\boldsymbol{G}_f + \boldsymbol{G}_g) & \overline{\overline{\boldsymbol{T}}}_f + \overline{\overline{\boldsymbol{T}}}_g \end{pmatrix}. \tag{46}$$

Furthermore, the gravitoelectromagnetic field variables in $T_g^{\mu\nu}$ are given in terms of the scalar ϕ_g and vector $\boldsymbol{A}_g$ potentials by the definitions (5), here repeated for convenience:

$$\boldsymbol{E}_g = -\boldsymbol{\nabla}\phi_g - \partial \boldsymbol{A}_g/\partial t \text{ and } \boldsymbol{B}_g = \boldsymbol{\nabla} \times \boldsymbol{A}_g. \tag{47}$$

In this form the field equation for the metric perturbation constitutes, combined with the harmonic gauge and the conservation of specific entropy, a closed set of equations for describing the gravitational field and the fluid dynamics, while preserving momentum-energy conservation in flat space.

One must keep in mind that in general relativity the exchange of energy and momentum is allowed only between matter and non-gravitational fields represented by $T_{\mu\nu}$. This exchange is not allowed between matter and the gravitational field itself, which is governed by the exact Einstein's field equations. In the present extended gravitoelectromagnetic formulation it is assumed that $\widetilde{h}_{\mu\nu}$ is generated by the total densities of energy and momentum in the flat-space, fluid-field system. This non-standard assumption implies that mass and energy share the same role in establishing gravity, and will be further clarified in Subsection III A. The difference between the two formulations involves second order terms in the perturbation $\widetilde{h}_{\mu\nu}$, which can be possibly corrected by adjusting boundary conditions. Alternatively, one may consider that in the limit of very small metric perturbations ($\left|\widetilde{h}_{\mu\nu}\right| \to 0$) the fluid dynamics in flat space is described by the gravitoelectromagnetic equations listed in Section II. The usual Newtonian limit corresponds to weak-field, quasi-stationary and non-relativistic motions, while the gravitoelectromagnetic description includes the weak-field effects of fast-varying, relativistic mass currents.

Note: The transformation

$$\partial^\nu h_{\mu\nu} \to \left(\partial^\nu \widetilde{h}_{\mu\nu}\right)' = \partial^\nu \widetilde{h}_{\mu\nu} - \Box^2 \xi_\mu \tag{48}$$

shows that the harmonic gauge $\partial^\nu \widetilde{h}_{\mu\nu} = 0$ is preserved by a coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ with

$$\Box^2 \xi_\mu = 0. \tag{49}$$

If $\Box^2 \xi_\mu = 0$, then also $\Box^2 \xi_{\mu\nu} = 0$, where

$$\xi_{\mu\nu} = (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) - \eta_{\mu\nu}\partial_\rho \xi^\rho \tag{50}$$

because the flat-space d'Alembertian $\Box^2$ commutes with the ordinary derivatives ∂_μ. Then, the transformation

$$h_{\mu\nu} \to \widetilde{h}'_{\mu\nu} = \widetilde{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho) \tag{51}$$

shows that solutions of the homogeneous equation $\Box^2 \widetilde{h}_{\mu\nu} = 0$ (true vacuum solutions) can be modified by the subtraction of the arbitrary functions ξ_μ, which satisfy the same equation $\Box^2 \xi_{\mu\nu} = 0$. This means that the four arbitrary functions $\xi_{\mu\nu}$ can be chosen so as to impose four conditions on the homogeneous solutions $\widetilde{h}_{\mu\nu}$. These gauge conditions can not be imposed on the linearized Einstein equation either inside the fluid source or in the near-field region, where $T_{\mu\nu} \neq 0$ and $\Box^2 \widetilde{h}_{\mu\nu} \neq 0$. Nevertheless, homogeneous solutions can always be added inside the source, representing gravitational radiation coming from infinity. As will be seen in Subsection III B, two of these gauge conditions can be applied in vacuum, both in the near- and far-field regions, but the application of the full set of four gauge conditions is restricted to the far-field (radiation) region.

A. Metric perturbation in the fluid region

The metric tensor perturbation inside the fluid can be written in terms of an effective scalar potential ϕ_{eff} and an effective vector potential $\boldsymbol{A}_{\text{eff}}$ in the following form

$$\widetilde{h}_{\mu\nu} = 2 \begin{pmatrix} -\phi_{\text{eff}}/c^2 & \boldsymbol{A}_{\text{eff}}/c \\ \boldsymbol{A}_{\text{eff}}/c & -\phi_{\text{eff}}\overline{\overline{\mathbf{I}}}/c^2 + \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}/c^2 \end{pmatrix}. \tag{52}$$

Here the scalar ϕ_{eff}, the vector $\boldsymbol{A}_{\text{eff}}$ and the symmetric dyadic $\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}$ represent the ten components of the symmetric tensor $\widetilde{h}_{\mu\nu}$. The harmonic gauge gives the following relations between the components of $\widetilde{h}_{\mu\nu}$:

$$\begin{aligned} \partial^\nu \widetilde{h}_{\mu\nu} &= \left(-\frac{1}{c}\frac{\partial}{\partial t} \ , \ \boldsymbol{\nabla} \right) \cdot 2 \begin{pmatrix} -\phi_{\text{eff}}/c^2 & \boldsymbol{A}_{\text{eff}}/c \\ \boldsymbol{A}_{\text{eff}}/c & -\phi_{\text{eff}}\overline{\overline{\boldsymbol{I}}}/c^2 + \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}/c^2 \end{pmatrix}^T \\ &= 2 \begin{pmatrix} \dfrac{1}{c}\left(\dfrac{1}{c^2}\dfrac{\partial \phi_{\text{eff}}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}_{\text{eff}}\right) \\ \dfrac{1}{c^2}\left(-\dfrac{\partial \boldsymbol{A}_{\text{eff}}}{\partial t} - \boldsymbol{\nabla}\phi_{\text{eff}} + \boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}\right) \end{pmatrix} = 0. \end{aligned} \tag{53}$$

The time- and space-like components of this condition give

$$\begin{cases} \boldsymbol{\nabla}\cdot\boldsymbol{A}_{\text{eff}} + \dfrac{1}{c^2}\dfrac{\partial \phi_{\text{eff}}}{\partial t} = 0 \\ \boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} = \boldsymbol{\nabla}\phi_{\text{eff}} + \dfrac{\partial \boldsymbol{A}_{\text{eff}}}{\partial t} \end{cases} \tag{54}$$

Taking the curl and the divergence of the space-like component one obtains

$$\begin{aligned} \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}\right) &= \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times\boldsymbol{A}_{\text{eff}}\right), \\ \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}\right) &= \boldsymbol{\nabla}^2\phi_{\text{eff}} + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\text{eff}}\right) = \boldsymbol{\nabla}^2\phi_{\text{eff}} - \frac{1}{c^2}\frac{\partial^2\phi_{\text{eff}}}{\partial t^2} = \Box^2\phi_{\text{eff}}. \end{aligned} \tag{55}$$

The above relations are used to define the effective gravitoelectric and gravitomagnetic fields either in terms of the tensor $\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}$, or in terms of the potentials ϕ_{eff} and $\boldsymbol{A}_{\text{eff}}$:

$$\begin{aligned} \boldsymbol{E}_{\text{eff}} &= -\boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} = -\boldsymbol{\nabla}\phi_{\text{eff}} - \frac{\partial \boldsymbol{A}_{\text{eff}}}{\partial t}, \\ \frac{\partial \boldsymbol{B}_{\text{eff}}}{\partial t} &= \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}\right) = \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times\boldsymbol{A}_{\text{eff}}\right). \end{aligned} \tag{56}$$

The effective gravitoelectromagnetic field satisfies the consistency relations

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{E}_{\text{eff}} &= -\frac{\partial \boldsymbol{B}_{\text{eff}}}{\partial t} && \text{effective GEM Faraday's law} \\ \boldsymbol{\nabla}\cdot\boldsymbol{B}_{\text{eff}} &= 0 && \text{effective GM Gauss's law} \end{aligned} \tag{57}$$

Now, the field equation yields
(***i***) For $\mu = 0$ and $\nu = 0$:

$$\Box^2\widetilde{h}_{00} = -\frac{16\pi G}{c^4}T_{00} \quad \therefore\ \Box^2\phi_{\text{eff}} = \frac{8\pi G}{c^2}\left(U_f + U_g\right). \tag{58}$$

(***ii***) For $\mu = 0$ and $\nu = i$:

$$\Box^2\widetilde{h}_{0i} = -\frac{16\pi G}{c^4}T_{0i} \quad \therefore\ \Box^2\boldsymbol{A}_{\text{eff}} = -\frac{8\pi G}{c^2}\left(\boldsymbol{G}_f + \boldsymbol{G}_g\right). \tag{59}$$

(***iii***) For $\mu = i$ and $\nu = j$:

$$\Box^2\widetilde{h}_{ij} = -\frac{16\pi G}{c^4}T_{ij} \quad \therefore\ \Box^2\left(\phi_{\text{eff}}\overline{\overline{\boldsymbol{I}}} - \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}\right) = \frac{8\pi G}{c^2}\left(\overline{\overline{\boldsymbol{T}}}_f + \overline{\overline{\boldsymbol{T}}}_g\right). \tag{60}$$

Thus the effective fields satisfy the set of inhomogeneous wave equations

$$\begin{cases} \Box^2\phi_{\text{eff}} = \dfrac{8\pi G}{c^2}\left(U_f + U_g\right) = 8\pi G\left[\gamma\rho + \left(\dfrac{1}{\gamma_A - 1} + \beta^2\right)\dfrac{\gamma^2 p}{c^2} + \dfrac{U_g}{c^2}\right] \\ \Box^2\boldsymbol{A}_{\text{eff}} = -\dfrac{8\pi G}{c^2}\left(\boldsymbol{G}_f + \boldsymbol{G}_g\right) = -\dfrac{8\pi G}{c^2}\left[\left(\gamma\rho + \dfrac{\gamma_A}{\gamma_A - 1}\dfrac{\gamma^2 p}{c^2}\right)\boldsymbol{u} + \dfrac{\boldsymbol{S}_g}{c^2}\right] \\ \Box^2\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} = \dfrac{8\pi G}{c^2}\left[\left(U_f + U_g\right)\overline{\overline{\boldsymbol{I}}} - \left(\overline{\overline{\boldsymbol{T}}}_f + \overline{\overline{\boldsymbol{T}}}_g\right)\right] \end{cases} \tag{61}$$

which can always be integrated inverting the d'Alembertian operator. Accordingly, the effective scalar potential ϕ_{eff} is given in terms of the total fluid and gravitoelectromagnetic energy density, $U = U_f + U_g$, by

$$\phi_{\text{eff}}(\boldsymbol{r},t) = -\frac{2G}{c^2}\int \frac{U(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r', \tag{62}$$

where $t' = t - |\boldsymbol{r}-\boldsymbol{r}'|/c$ is the retarded time. Similarly, the effective vector potential $\boldsymbol{A}_{\text{eff}}$ and the tensor $\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}$ are given in terms of the total momentum density $\boldsymbol{G} = \boldsymbol{G}_f + \boldsymbol{G}_g$ and of the total stress tensor $\overline{\overline{\boldsymbol{T}}} = \overline{\overline{\boldsymbol{T}}}_f + \overline{\overline{\boldsymbol{T}}}_g$ by, respectively,

$$\begin{aligned} \boldsymbol{A}_{\text{eff}}(\boldsymbol{r},t) &= \frac{2G}{c^2}\int \frac{\boldsymbol{G}(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r', \\ \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}(\boldsymbol{r},t) &= \phi_{\text{eff}}(\boldsymbol{r},t)\overline{\overline{\boldsymbol{I}}} + \frac{2G}{c^2}\int \frac{\overline{\overline{\boldsymbol{T}}}(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r'. \end{aligned} \tag{63}$$

In this way the metric tensor perturbations can be determined in terms of the fluid variables ρ, p and $\boldsymbol{u}$, and of the gravitoelectromagnetic field variables ϕ_g and $\boldsymbol{A}_g$. The fluid and field variables satisfy the equation of momentum conservation, the energy conservation equation, the equation of state for isentropic flow, and the equations of Poisson and Ampère. In other words, the gravitoelectromagnetic field variables are determined in terms of consistent distributions of mass and current by means of the Maxwell source equations; and the metric tensor perturbations depend both on the mass and field distributions. The harmonic gauge condition $\partial^\nu \widetilde{h}_{\mu\nu} = 0$ is equivalent, at least up to first order in $|h_{\mu\nu}|$, to the energy-momentum conservation equation $\partial_\nu T^{\mu\nu} = 0$, which must be consistently solved using the equations listed in Section II. Note also that an order of magnitude comparison of the field equations for the metric perturbations with dominant mass density contribution gives

$$\begin{aligned} \Box^2 \widetilde{h}_{00} &= -\frac{16\pi G}{c^2}\left[\gamma\rho + \left(\frac{1}{\gamma_A - 1} + \beta^2\right)\frac{\gamma^2 p}{c^2} + \frac{U_g}{c^2}\right] \sim -\frac{16\pi G}{c^2}\gamma\rho \sim 1, \\ \Box^2 \widetilde{h}_{0i} &= -\frac{16\pi G}{c^3}\left[\left(\gamma\rho + \frac{\gamma_A}{\gamma_A - 1}\frac{\gamma^2 p}{c^2}\right)\boldsymbol{u} + \frac{\boldsymbol{S}_g}{c^2}\right]_{0i} \sim -\frac{16\pi G}{c^2}\gamma\rho\frac{\boldsymbol{u}_i}{c} \sim \beta, \\ \Box^2 \widetilde{h}_{ij} &= -\frac{16\pi G}{c^4}\left[\gamma\rho\boldsymbol{u}\boldsymbol{u} + \left(\overline{\overline{\boldsymbol{I}}} + \frac{\gamma_A}{\gamma_A - 1}\gamma^2\frac{\boldsymbol{u}\boldsymbol{u}}{c^2}\right)p + \overline{\overline{\boldsymbol{T}}}_g\right]_{ij} \sim -\frac{16\pi G}{c^2}\gamma\rho\frac{\boldsymbol{u}_i\boldsymbol{u}_j}{c^2} \\ &\qquad \sim \beta^2. \end{aligned} \tag{64}$$

These relations clearly show that for dominant matter the time-space components are related to the relativistic mass currents, and the space-space components are related to the fluid stresses.

In this form, by keeping the field contributions, the metric tensor perturbations inside the fluid source can be solved consistently with the fluid motion in the gravitational field itself. Physically, the first-order Minkowski approximation is applied in the space between the fluid particles (dust) where the motion is governed by mean Vlasov fields produced by the same point-like particles. Otherwise, the fluid motion must be governed by extraneous non-gravitational fields, or simply introduced in the form of given sources, not determined consistently. The same situation exists in the classic problem of linear electromagnetism, where either the fields are calculated in terms of given sources, or the motion of the particles is calculated for given fields. It is clear also that the field contributions to the metric are of the order of G^2.

As will be seen in the next subsection, in the vacuum region the metric perturbations depend on the field components only, which are determined in terms of the internal motion of the fluid sources. Accordingly, one may consider the fluid itself as a distribution of particles (dust) immersed in a "flat" vacuum (excepting boundary regions). Metric perturbations due to microscopic motion cancel out, so that only macroscopic fluid motion produces significant gravitational perturbations.

B. Metric perturbation in the vacuum region

In the vacuum region near a mass distribution the metric tensor perturbation depends on the gravitoelectromagnetic field components only. The metric perturbation can be written as

$$\widetilde{h}_{\mu\nu} = 2\begin{pmatrix} -\phi_g/c^2 & \boldsymbol{A}_g/c \\ \boldsymbol{A}_g/c & -\phi_g\overline{\overline{\boldsymbol{I}}}/c^2 + \overline{\overline{\boldsymbol{\psi}}}_g/c^2 \end{pmatrix}, \tag{65}$$

and the harmonic gauge leads to

$$\partial^\nu \widetilde{h}_{\mu\nu} = 0 \quad \begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{A}_g + \dfrac{1}{c^2}\dfrac{\partial \phi_g}{\partial t} = 0 & \text{Lorenz gauge} \\ \boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g = \boldsymbol{\nabla}\phi_g + \dfrac{\partial \boldsymbol{A}_g}{\partial t} = -\boldsymbol{E}_g & \text{GEM field} \end{cases} \tag{66}$$

where the second equation serves to confirm the definition of the gravitoelectromagnetic field $\boldsymbol{E}_g$. Furthermore

$$\begin{aligned} \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g\right) &= \frac{\partial}{\partial t}\left(\boldsymbol{\nabla} \times \boldsymbol{A}_g\right) = \frac{\partial \boldsymbol{B}_g}{\partial t}, \\ \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g\right) &= \boldsymbol{\nabla}^2\phi_g + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla} \cdot \boldsymbol{A}_g\right) = \boldsymbol{\nabla}^2\phi_g - \frac{1}{c^2}\frac{\partial^2 \phi_g}{\partial t^2} = \Box^2\phi_g. \end{aligned} \tag{67}$$

The harmonic gauge $\partial^\nu \widetilde{h}_{\mu\nu} = 0$ relates $\overline{\overline{\boldsymbol{\psi}}}_g$ to the gravitoelectromagnetic field according to the previous definition (cf. equation 5)

$$\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g = \boldsymbol{\nabla}\phi_g + \frac{\partial \boldsymbol{A}_g}{\partial t} = -\boldsymbol{E}_g, \tag{68}$$

and gives a straightforward relation in a region free of mass sources

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g\right) = \Box^2\phi_g = -\boldsymbol{\nabla} \cdot \boldsymbol{E}_g = 0. \tag{69}$$

In the Lorenz gauge the scalar potential ϕ_g and the vector potential $\boldsymbol{A}_g$ satisfy the homogeneous wave equations (cf. equation (18)):

$$\Box^2\phi_g = 0 \qquad \text{and} \;\; \Box^2\boldsymbol{A}_g = 0. \tag{70}$$

Hence

$$\Box^2\widetilde{h}_{\mu\nu} = 2\begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & \Box^2\overline{\overline{\boldsymbol{\psi}}}_g/c^2 \end{pmatrix}. \tag{71}$$

The symmetric tensor $\overline{\overline{\boldsymbol{\psi}}}_g$ may be formed by the juxtaposition of two vectors $\boldsymbol{a}$ and $\boldsymbol{e} = |\boldsymbol{e}|\,\widehat{\boldsymbol{e}}$, where $\widehat{\boldsymbol{e}}$ denotes the unit vector in the direction of $\boldsymbol{e}$:

$$\begin{aligned} \overline{\overline{\boldsymbol{\psi}}}_g &= \frac{\boldsymbol{ae} + \boldsymbol{ea}}{2} = a_\parallel \widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + |\boldsymbol{e}|\left(\frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right), \\ \boldsymbol{a} \cdot \widehat{\boldsymbol{e}} &= a_\parallel = \overline{\overline{\boldsymbol{\psi}}}_g : \widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} = \overline{\overline{\boldsymbol{\psi}}}_g : \overline{\overline{\boldsymbol{I}}}, \\ \boldsymbol{a}_\perp \cdot \widehat{\boldsymbol{e}} &= 0, \\ \boldsymbol{a}_\perp &= \frac{2}{|\boldsymbol{e}|}\left(\overline{\overline{\boldsymbol{\psi}}}_g \cdot \widehat{\boldsymbol{e}} - a_\parallel\widehat{\boldsymbol{e}}\right) = \frac{2}{|\boldsymbol{e}|}\left(\widehat{\boldsymbol{e}} \cdot \overline{\overline{\boldsymbol{\psi}}}_g - a_\parallel\widehat{\boldsymbol{e}}\right). \end{aligned} \tag{72}$$

The trace of the perturbation $\widetilde{h}_{\mu\nu}$ in vacuum is

$$\begin{aligned} \widetilde{h} &= \eta^{\mu\nu}\widetilde{h}_{\mu\nu} = \begin{pmatrix} -1 & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\overline{\boldsymbol{I}}} \end{pmatrix} : 2\begin{pmatrix} -\phi_g/c^2 & \boldsymbol{A}_g/c \\ \boldsymbol{A}_g/c & -\phi_g\overline{\overline{\boldsymbol{I}}}/c^2 + \overline{\overline{\boldsymbol{\psi}}}_g/c^2 \end{pmatrix} \\ &= \frac{2}{c^2}\left(\overline{\overline{\boldsymbol{\psi}}}_g : \overline{\overline{\boldsymbol{I}}} - 2\phi_g\right) = \frac{2}{c^2}\left(a_\parallel - 2\phi_g\right). \end{aligned} \tag{73}$$

Thus

$$\Box^2\widetilde{h} = \frac{2}{c^2}\Box^2\left(a_\parallel - 2\phi_g\right) = \frac{2}{c^2}\Box^2 a_\parallel. \tag{74}$$

One verifies that $\Box^2\widetilde{h}_{\mu\nu}$ has vanishing time-time and vanishing mixed time-space components. But the space-space components

$$\Box^2\widetilde{h}_{ij} = 2\Box^2\overline{\overline{\boldsymbol{\psi}}}_g/c^2 = \Box^2\left(\boldsymbol{ae} + \boldsymbol{ea}\right)/c^2, \tag{75}$$

which correspond to the gravitational waves contribution, do not vanish in general.

As pointed out in the note at the end of the introductory Section III, the harmonic gauge $\partial^\nu \widetilde{h}_{\mu\nu} = 0$ is maintained by a coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ with

$$\Box^2 \xi_\mu = 0, \tag{76}$$

and the solutions of the homogeneous equation $\Box^2 \widetilde{h}_{\mu\nu} = 0$ can be transformed as

$$h_{\mu\nu} \to \widetilde{h}'_{\mu\nu} = \widetilde{h}_{\mu\nu} - \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho\right), \tag{77}$$

where the function

$$\xi_{\mu\nu} = \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right) - \eta_{\mu\nu} \partial_\rho \xi^\rho \tag{78}$$

satisfies the same equation $\Box^2 \xi_{\mu\nu} = 0$. Therefore, the solutions of $\Box^2 \widetilde{h}_{\mu\nu} = 0$ can be modified by the subtraction of the functions $\xi_{\mu\nu}$, that is, a total of four arbitrary functions ξ_μ can be chosen so as to impose four conditions on the homogeneous solutions $\widetilde{h}_{\mu\nu}$. However, in the present problem only two functions defined by

$$\begin{aligned} \xi_{00} &= \left(\partial_0 \xi_0 + \partial_0 \xi_0\right) - \eta_{00} \partial_\rho \xi^\rho = 2\partial_0 \xi_0 + \partial_\rho \xi^\rho, \\ \xi_{0i} &= \xi_{i0} = \left(\partial_0 \xi_i + \partial_i \xi_0\right) - \eta_{0i} \partial_\rho \xi^\rho = \left(\partial_0 \xi_i + \partial_i \xi_0\right) - \partial_\rho \xi^\rho, \end{aligned} \tag{79}$$

which satisfy the equations

$$\Box^2 \xi_{00} = 0 \qquad \text{and} \quad \Box^2 \xi_{0i} = 0, \tag{80}$$

can be safely subtracted from $\Box^2 \widetilde{h}_{\mu\nu} = 0$ without modifying the solution in the vacuum region. Taking into account this gauge freedom one can impose the following set of conditions on the metric tensor in vacuum:

$$\begin{array}{rl} \widetilde{h} = 0 & \text{traceless gauge condition} \\ |\boldsymbol{e}| = 1 & \text{normalization of the reference direction} \end{array} \tag{81}$$

Therefore, in the vacuum region the stress tensor $\overline{\overline{\boldsymbol{\psi}}}_g$ can be represented by

$$\overline{\overline{\boldsymbol{\psi}}}_g = a_\parallel \widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \frac{\boldsymbol{a}_\perp \widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2} = 2\phi_g \widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \frac{\boldsymbol{a}_\perp \widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}, \tag{82}$$

so that

$$\widetilde{h} = \frac{2}{c^2}\left(\overline{\overline{\boldsymbol{\psi}}}_g : \overline{\overline{\boldsymbol{I}}} - 2\phi_g\right) = \frac{2}{c^2}\left(a_\parallel - 2\phi_g\right) = 0 \implies a_\parallel = 2\phi_g, \tag{83}$$

and $\widehat{\boldsymbol{e}}$ establishes a reference direction (direction of propagation of the gravitational wave). The metric perturbation in the vacuum region becomes

$$\widetilde{h}_{\mu\nu} = 2 \begin{pmatrix} -\dfrac{\phi_g}{c^2} & \dfrac{\boldsymbol{A}_g}{c} \\ \dfrac{\boldsymbol{A}_g}{c} & -\dfrac{\phi_g}{c^2}\left(\overline{\overline{\boldsymbol{I}}} - 2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}\right) + \dfrac{\boldsymbol{a}_\perp \widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2c^2} \end{pmatrix}. \tag{84}$$

As stated previously, the harmonic gauge $\partial^\nu \widetilde{h}_{\mu\nu} = 0$ corresponds to the Lorenz gauge for ϕ_g and $\boldsymbol{A}_g$, and to a relation between $\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g$ and $\boldsymbol{E}_g$. The Lorenz gauge for ϕ_g and $\boldsymbol{A}_g$ can be replaced by the equivalent condition $\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g\right) = 0$ so that

$$\partial^\nu \widetilde{h}_{\mu\nu} = 0 \quad \begin{cases} \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g\right) = \boldsymbol{\nabla} \cdot \left[\boldsymbol{\nabla} \cdot \left(2\phi_g \widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \dfrac{\boldsymbol{a}_\perp \widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right)\right] = 0 \\ \boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\psi}}}_g = \boldsymbol{\nabla} \cdot \left(2\phi_g \widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \dfrac{\boldsymbol{a}_\perp \widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right) = \boldsymbol{\nabla}\phi_g + \dfrac{\partial \boldsymbol{A}_g}{\partial t} = -\boldsymbol{E}_g \end{cases} \tag{85}$$

Besides ϕ_g the symmetric tensor $\overline{\overline{\boldsymbol{\psi}}}_g$ is characterized by four components represented by $\boldsymbol{a}_\perp$ and $\widehat{\boldsymbol{e}}$ (the unit vector $\widehat{\boldsymbol{e}}$ is described by two direction cosines). These components can, in principle, be determined in terms of ϕ_g and $\boldsymbol{A}_g$

using the four equations provided by the harmonic gauge for $\widetilde{h}_{\mu\nu}$, with $\boldsymbol{E}_g$ acting as a driving field. Moreover, ϕ_g and $\boldsymbol{A}_g$ satisfy the homogeneous wave equations

$$\Box^2\phi_g = 0 \qquad \text{and} \quad \Box^2\boldsymbol{A}_g = 0, \tag{86}$$

which allow the solution of the problem in the vacuum region for given initial conditions at the free-boundary fluid-vacuum interface. The metric gravitational waves are generated inside the fluid by the gravitoelectromagnetic waves (fluid oscillations), due to the coupling provided by the metric perturbations. Inside the fluid the metric gravitational waves are related to the gravitoelectromagnetic waves in an intricate way. Outside the fluid, but near the fluid-vacuum interface, there are strong capacitive and inductive effects from the internal mass and mass current distributions that affect both the non-propagating and propagating gravitoelectromagnetic components. Conversely, the gravitoelectromagnetic field back-pressure affects the position of the free-boundary between the fluid and vacuum. Also, the propagation of the radiating gravitoelectromagnetic components is affected by the field near the interface. One may say that the gravitational waves suffer refraction or diffraction effects in the near-field region. Far from the interface the gravitational waves become nearly plane waves transmitting net energy at large distances from the source. For a fixed orientation $\widehat{\boldsymbol{e}} = \widehat{\boldsymbol{z}}$ the stress tensor in the far-field region can be written in matrix form

$$\overline{\overline{\boldsymbol{\psi}}}_g = 2\phi_g\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2} = \begin{pmatrix} 0 & 0 & \boldsymbol{a}_\perp\cdot\widehat{\boldsymbol{x}}/2 \\ 0 & 0 & \boldsymbol{a}_\perp\cdot\widehat{\boldsymbol{y}}/2 \\ \boldsymbol{a}_\perp\cdot\widehat{\boldsymbol{x}}/2 & \boldsymbol{a}_\perp\cdot\widehat{\boldsymbol{y}}/2 & 2\phi_g \end{pmatrix}. \tag{87}$$

The components of $\boldsymbol{a}_\perp$ are related to the components of the gravitoelectromagnetic force $\partial\boldsymbol{A}_g/\partial t$ which, in the high frequency regime, are given in terms of the mass current density fluctuations in the source. The equations for $\overline{\overline{\boldsymbol{\psi}}}_g$ describe the propagation of gravitational waves at the speed of light in the vacuum region. These equations must be simultaneously solved with the equations for ϕ_g and $\boldsymbol{A}_g$ in order to determine the four components $\boldsymbol{a}_\perp$ and $\widehat{\boldsymbol{e}}$ of $\overline{\overline{\boldsymbol{\psi}}}_g$ which satisfy the boundary conditions at the fluid-vacuum interface.

Recall that the wave equation in the vacuum region gives

$$\Box^2\widetilde{h}_{\mu\nu} = \begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & \frac{2}{c^2}\Box^2\underbrace{\left(2\phi_g\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right)}_{\overline{\overline{\boldsymbol{\psi}}}_g} \end{pmatrix}. \tag{88}$$

One verifies that the homogeneous wave equation $\Box^2\widetilde{h}_{\mu\nu} = 0$ can be fully satisfied if ϕ_g, $\boldsymbol{a}_\perp$ and $\widehat{\boldsymbol{e}}$ are governed by the wave equation

$$\Box^2\overline{\overline{\boldsymbol{\psi}}}_g = \Box^2\left(2\phi_g\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right) = 0, \tag{89}$$

which gives additional constraints for $\boldsymbol{a}_\perp$ and $\widehat{\boldsymbol{e}}$ taking notice that $\Box^2\phi_g = 0$. In the far-field region, for a fixed direction $\widehat{\boldsymbol{e}}$ (which can be identified as the propagation direction of the gravitational wave), this reduces to a wave equation for $\boldsymbol{a}_\perp$ (with two independent polarizations)

$$\Box^2\boldsymbol{a}_\perp = 0. \tag{90}$$

However, the additional constraints imposed by $\Box^2\overline{\overline{\boldsymbol{\psi}}}_g = 0$ cannot in general be satisfied in the near-field region. Only in the far-field region the gravitational wave detaches from the source and propagates independently. The additional condition $\Box^2\overline{\overline{\boldsymbol{\psi}}}_g \sim 0$ can be used to formally separate the near- and far-field regions.

If gravitational waves are neglected, that is, $\boldsymbol{a}_\perp \sim 0$, the gauge freedom allows to impose the following set of four conditions on the metric tensor in vacuum:

$$\begin{aligned} \widetilde{h} &= 0 \quad \text{traceless gauge condition} \\ \partial\boldsymbol{A}_g/\partial t &\sim 0 \quad \text{non-radiating regime} \end{aligned} \tag{91}$$

This case corresponds to the low-frequency regime with the induction gravitoelectromagnetic fields vanishing in the far-field region. Note that the gravitoelectromotive force $-\partial\boldsymbol{A}_g/\partial t$ is responsible for the gravitoelectromagnetic field associated with metric gravitational waves. Note also that the non-radiating regime condition $\partial\boldsymbol{A}_g/\partial t \sim 0$ breaks the connection of the gravitational waves with the source. In this case the gravitoelectromagnetic fields $\boldsymbol{E}_g$ and $\boldsymbol{B}_g$

represent induction fields that do not radiate according to the Lorenz gauge condition (note that the fields are not necessarily static, but slowly varying only):

$$\begin{cases} \boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_g & = \boldsymbol{\nabla}\phi_g + \dfrac{\partial \boldsymbol{A}_g}{\partial t} = -\boldsymbol{E}_g & \Longrightarrow \boldsymbol{E}_g \sim -\boldsymbol{\nabla}\phi_g \\ \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_g\right) & = \dfrac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times\boldsymbol{A}_g\right) = \dfrac{\partial \boldsymbol{B}_g}{\partial t} & \Longrightarrow \dfrac{\partial \boldsymbol{B}_g}{\partial t} \sim 0 \\ \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_g\right) & = \boldsymbol{\nabla}^2\phi_g + \dfrac{\partial}{\partial t}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}_g\right) = \Box^2\phi_g = 0 & \Longrightarrow \boldsymbol{\nabla}^2\phi_g \sim 0 \end{cases} \tag{92}$$

In general, the gravitoelectromagnetic field in vacuum must satisfy the boundary conditions at the fluid-vacuum interface. Assuming that there are neither a surface mass density, $\sigma = 0$, nor a surface mass current density, $\boldsymbol{K} = 0$, the metric tensor perturbations must be continuous at the fluid-vacuum interface:

$$\begin{aligned} \phi_g\big|_s &= \phi_{\text{eff}}\big|_s\,, \\ \boldsymbol{A}_g\big|_s &= \boldsymbol{A}_{\text{eff}}\big|_s\,, \\ \overline{\overline{\boldsymbol{\psi}}}_g\Big|_s &= \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}\Big|_s\,. \end{aligned} \tag{93}$$

They must also satisfy the conditions at large distances from the mass source, where only gravitoelectromagnetic and gravitational waves propagate.

The scalar potential ϕ_g and the vector potential $\boldsymbol{A}_g$ satisfy the wave equations in vacuum $\Box^2\phi_g = 0$ and $\Box^2\boldsymbol{A}_g = 0$. Thus the scalar potential ϕ_g and the vector potential $\boldsymbol{A}_g$ near the fluid-vacuum interface are given in terms of the retarded mass and mass current distributions inside the source

$$\begin{cases} \phi_g\left(\boldsymbol{r},t\right) = -G\displaystyle\int \frac{\rho\left(\boldsymbol{r}',t'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r' \\ \boldsymbol{A}_g\left(\boldsymbol{r},t\right) = -\dfrac{G}{c^2}\displaystyle\int \frac{\boldsymbol{j}\left(\boldsymbol{r}',t'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r' \end{cases} \tag{94}$$

where $t' = t - \left|\boldsymbol{r}-\boldsymbol{r}'\right|/c$ is the retarded time. The gravitoelectromagnetic fields are determined, in the vacuum region, by both the mass and mass current distributions inside the source, and by the conditions of field continuity at the fluid-vacuum interface. If gravitational waves are neglected the gravitoelectromagnetic fields become induction fields (low-frequency regime). In this case the scalar potential ϕ_g is given by the instantaneous Coulomb's law. If gravitational waves are neglected the vector potential $\boldsymbol{A}_g$ in the vacuum region is given by the instantaneous Biot-Savart law. The scalar or Newtonian potential ϕ_g, in particular, can be approximately determined by solutions of Laplace's equation that fit the boundary conditions at the fluid-vacuum interface ($\phi_g = \phi_{\text{eff}}$ at the boundary). Nevertheless, the fields are governed by different equations in the regions inside and outside the source, with boundary conditions set at the source-vacuum interface.

IV. GEODESIC EQUATION

An event in a defined point in space and instant of time along a worldline is characterized by the coordinates $x^\mu(\tau)$ as a function of the proper time τ. The frame-independent space-time element is defined by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{95}$$

where $x^\mu = (ct, \boldsymbol{r})$. The events registered by an inertial clock occur all at the same place, so that $dx^i = 0$ and the corresponding proper time element $d\tau$ is

$$c^2 d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu. \tag{96}$$

Thus

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2. \tag{97}$$

In terms of the four-velocity $v^\mu = dx^\mu/d\tau$, this relation becomes

$$g_{\mu\nu}v^\mu v^\nu = -c^2. \tag{98}$$

The worldline of a test particle in a gravitational field is obtained extremizing the action

$$S = -mc^2 \int_{\tau_1}^{\tau_2} d\tau. \quad (99)$$

This leads to the geodesic equation for $x^\mu(\tau)$, which is written in compact form using the Christoffel symbol [10–12]:

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (100)$$

This is the equation of motion of a test particle according to general relativity. The Christoffel symbol is defined in terms of the metric as

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right). \quad (101)$$

Now, the space-time element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ in the weak field approximation is calculated in terms of the linearized metric tensor (40)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h} \ \text{, with} \ \left|\widetilde{h}_{\mu\nu}\right| \ll 1 \quad (102)$$

and the metric tensor perturbation $\widetilde{h}_{\mu\nu}$ is given by equation (52) in terms of the effective potentials inside the fluid:

$$\widetilde{h}_{\mu\nu} = 2 \begin{pmatrix} -\phi_{\text{eff}}/c^2 & \boldsymbol{A}_{\text{eff}}/c \\ \boldsymbol{A}_{\text{eff}}/c & -\phi_{\text{eff}} \overline{\overline{\boldsymbol{I}}}/c^2 + \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}/c^2 \end{pmatrix}. \quad (103)$$

The trace of $\widetilde{h} = -h$ is given by

$$\widetilde{h} = \frac{2}{c^2}\left(-2\phi_{\text{eff}} + \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} : \overline{\overline{\boldsymbol{I}}}\right) = \frac{2}{c^2}\left(\phi_{\text{eff}} + \frac{2G}{c^2}\int \frac{\overline{\overline{\boldsymbol{T}}}(\boldsymbol{r}', t') : \overline{\overline{\boldsymbol{I}}}}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r'\right), \quad (104)$$

and the metric tensor becomes

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} -1 & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\overline{I}} \end{pmatrix} + 2 \begin{pmatrix} -\phi_{\text{eff}}/c^2 & \boldsymbol{A}_{\text{eff}}/c \\ \boldsymbol{A}_{\text{eff}}/c & -\phi_{\text{eff}} \overline{\overline{\boldsymbol{I}}}/c^2 + \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}}/c^2 \end{pmatrix} \\ &\quad - \frac{1}{c^2}\left(-2\phi_{\text{eff}} + \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} : \overline{\overline{\boldsymbol{I}}}\right) \begin{pmatrix} -1 & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\overline{\boldsymbol{I}}} \end{pmatrix} \\ &= \begin{pmatrix} -1 - \frac{1}{c^2}\left(4\phi_{\text{eff}} - \overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} : \overline{\overline{\boldsymbol{I}}}\right) & \frac{2}{c}\boldsymbol{A}_{\text{eff}} \\ \frac{2}{c}\boldsymbol{A}_{\text{eff}} & \left(1 - \frac{1}{c^2}\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} : \overline{\overline{\boldsymbol{I}}}\right)\overline{\overline{\boldsymbol{I}}} + \frac{2}{c^2}\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} \end{pmatrix}. \end{aligned} \quad (105)$$

Thus

$$\begin{aligned} g_{00} &= -1 - \frac{\phi_{\text{eff}}}{c^2} + \frac{2G}{c^4}\int \frac{\overline{\overline{\boldsymbol{T}}}(\boldsymbol{r}', t') : \overline{\overline{\boldsymbol{I}}}}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r', \\ g_{i0} &= g_{0i} = \frac{2A_{\text{eff},i}}{c} = \frac{4G}{c^3}\int \frac{G_i(\boldsymbol{r}', t')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r', \\ g_{ij} &= g_{ji} = \left(1 - \frac{\phi_{\text{eff}}}{c^2} - \frac{2G}{c^4}\int \frac{\overline{\overline{\boldsymbol{T}}}(\boldsymbol{r}', t') : \overline{\overline{\boldsymbol{I}}}}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r'\right)\delta_{ij} + \frac{4G}{c^4}\int \frac{T_{ij}(\boldsymbol{r}', t')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r', \end{aligned} \quad (106)$$

where

$$\phi_{\text{eff}} = -\frac{2G}{c^2}\int \frac{U(\boldsymbol{r}', t')}{|\boldsymbol{r}-\boldsymbol{r}'|} d^3r'. \quad (107)$$

The space time element is simply

$$ds^2 = g_{00}c^2 dt^2 + 2g_{0i} c dt dx^i + g_{ij} dx^i dx^j. \quad (108)$$

The test particle motion is described by the geodesic equation:

$$\begin{cases} \sigma = 0 \; \rightarrow \; c\dfrac{d^2t}{d\tau^2} + \Gamma^0_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = 0 \\ \sigma = i \; \rightarrow \; \dfrac{d^2x^i}{d\tau^2} + \Gamma^i_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} = 0 \end{cases} \tag{109}$$

Considering $x^i = x^i\left[t\left(\tau\right)\right]$

$$\begin{aligned} \frac{d^2x^i}{dt^2} &= \frac{d\tau}{dt}\frac{d}{d\tau}\left(\frac{d\tau}{dt}\frac{dx^i}{d\tau}\right) = \left(\frac{d\tau}{dt}\right)^2\frac{d^2x^i}{d\tau^2} - \left(\frac{d\tau}{dt}\right)^3\frac{d^2t}{d\tau^2}\frac{dx^i}{d\tau} \\ &= -\left(\frac{d\tau}{dt}\right)^2\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \left(\frac{d\tau}{dt}\right)^3\frac{\Gamma^0_{\mu\nu}}{c}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{dx^i}{d\tau}. \end{aligned} \tag{110}$$

Hence, the equation of motion can be written as

$$\begin{aligned} \frac{d^2x^i}{dt^2} &= -\Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \frac{\Gamma^0_{\mu\nu}}{c}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{dx^i}{dt} \\ &= -c^2\Gamma^i_{00} - 2c\Gamma^i_{0j}\frac{dx^j}{dt} - \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} \\ &\quad + \frac{1}{c}\left(c^2\Gamma^0_{00} + 2c\Gamma^0_{0j}\frac{dx^j}{dt} + \Gamma^0_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right)\frac{dx^i}{dt}. \end{aligned} \tag{111}$$

The linearized Christoffel symbols are

$$\begin{aligned} \Gamma^\sigma_{\mu\nu} &\cong \frac{1}{2}\eta^{\sigma\rho}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}\right) \\ &= \frac{1}{2}\eta^{\sigma\rho}\left[\partial_\mu\left(\widetilde{h}_{\nu\rho} - \frac{1}{2}\eta_{\nu\rho}\widetilde{h}\right) + \partial_\nu\left(\widetilde{h}_{\rho\mu} - \frac{1}{2}\eta_{\rho\mu}\widetilde{h}\right) - \partial_\rho\left(\widetilde{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\widetilde{h}\right)\right]. \end{aligned} \tag{112}$$

In explicit time-time, time-space and space-space components

$$\begin{cases} \Gamma^i_{00} = \dfrac{1}{c}\dfrac{\partial}{\partial t}\widetilde{h}_{0i} - \dfrac{1}{2}\partial_i\left(\widetilde{h}_{00} + \dfrac{1}{2}\widetilde{h}\right) \\ \Gamma^i_{0j} = \dfrac{1}{2c}\dfrac{\partial}{\partial t}\left(\widetilde{h}_{ji} - \dfrac{1}{2}\delta_{ji}\widetilde{h}\right) + \dfrac{1}{2}\left(\partial_j\widetilde{h}_{i0} - \partial_i\widetilde{h}_{0j}\right) \\ \Gamma^i_{jk} = \dfrac{1}{2}\partial_j\left(\widetilde{h}_{ki} - \dfrac{1}{2}\delta_{ki}\widetilde{h}\right) + \dfrac{1}{2}\partial_k\left(\widetilde{h}_{ij} - \dfrac{1}{2}\delta_{ij}\widetilde{h}\right) - \dfrac{1}{2}\partial_i\left(\widetilde{h}_{jk} - \dfrac{1}{2}\delta_{jk}\overline{h}\right) \\ \Gamma^0_{00} = -\dfrac{1}{2c}\dfrac{\partial}{\partial t}\left(\widetilde{h}_{00} + \dfrac{1}{2}\widetilde{h}\right) \\ \Gamma^0_{0j} = -\dfrac{1}{2}\partial_j\left(\widetilde{h}_{00} + \dfrac{1}{2}\widetilde{h}\right) \\ \Gamma^0_{jk} = -\dfrac{1}{2}\left(\partial_j\widetilde{h}_{k0} + \partial_k\widetilde{h}_{0j}\right) + \dfrac{1}{2c}\dfrac{\partial}{\partial t}\left(\widetilde{h}_{jk} - \dfrac{1}{2}\delta_{jk}\overline{h}\right) \end{cases} \tag{113}$$

The metric tensor perturbations inside the fluid are given according to the definition (52) by

$$\begin{cases} \widetilde{h}_{00} = -\dfrac{2\phi_{\text{eff}}}{c^2} \\ \widetilde{h}_{0i} = \widetilde{h}_{i0} = \dfrac{2A_{\text{eff},i}}{c} \\ \widetilde{h}_{ij} = \widetilde{h}_{ji} = -\dfrac{2\phi_{\text{eff}}}{c^2}\delta_{ij} + \dfrac{2}{c^2}\psi_{\text{eff},ij} \\ \widetilde{h} = -\dfrac{4\phi_{\text{eff}}}{c^2} + \dfrac{2}{c^2}\overline{\overline{\boldsymbol{\psi}}}_{\text{eff}} : \overline{\overline{\boldsymbol{I}}} \end{cases} \tag{114}$$

so that

$$\left\{\begin{array}{rcl}\Gamma^i_{00} & = & \frac{2}{c^2}\frac{\partial}{\partial t}A_{\mathrm{eff},i}+\frac{1}{2c^2}\partial_i\left(4\phi_{\mathrm{eff}}-\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\\ \Gamma^i_{0j} & = & \frac{1}{2c^3}\frac{\partial}{\partial t}\left[2\psi_{\mathrm{eff},ji}-\left(\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\delta_{ji}\right]-\frac{1}{c}\left(\partial_iA_{\mathrm{eff},j}-\partial_jA_{\mathrm{eff},i}\right)\\ \Gamma^i_{jk} & = & \frac{1}{2c^2}\partial_j\left[2\psi_{\mathrm{eff},ki}-\left(\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\delta_{ki}\right]+\frac{1}{2c^2}\partial_k\left[2\psi_{\mathrm{eff},ij}-\left(\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\delta_{ij}\right]\\ & & -\frac{1}{2c^2}\partial_i\left[2\psi_{\mathrm{eff},jk}-\left(\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\delta_{jk}\right]\\ \Gamma^0_{00} & = & \frac{1}{2c^3}\frac{\partial}{\partial t}\left(4\phi_{\mathrm{eff}}-\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\\ \Gamma^0_{0j} & = & \frac{1}{2c^2}\partial_j\left(4\phi_{\mathrm{eff}}-\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\\ \Gamma^0_{jk} & = & \frac{1}{2c^3}\frac{\partial}{\partial t}\left[2\psi_{\mathrm{eff},jk}-\left(\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\delta_{jk}\right]-\frac{1}{c}\left(\partial_jA_{\mathrm{eff},k}+\partial_kA_{\mathrm{eff},j}\right)\end{array}\right. \tag{115}$$

Neglecting the second and third-order corrections in the test particle velocity (non relativistic approximation for the test particle only – the fluid velocity remains relativistic in general):

$$\begin{array}{rcl}\frac{d^2x^i}{dt^2} & \cong & -c^2\Gamma^i_{00}-2c\Gamma^i_{0j}\frac{dx^j}{dt}+c\Gamma^0_{00}\frac{dx^i}{dt}+\ldots\\ & = & -\frac{1}{2}\partial_i\left(4\phi_{\mathrm{eff}}-\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)-2\frac{\partial A_{\mathrm{eff},i}}{\partial t}+2\left(\partial_iA_{\mathrm{eff},j}-\partial_jA_{\mathrm{eff},i}\right)\frac{dx^j}{dt}\\ & & +\frac{1}{2c^2}\frac{\partial}{\partial t}\left(4\phi_{\mathrm{eff}}-\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\frac{dx^i}{dt}-\frac{1}{c^2}\frac{\partial}{\partial t}\left[2\psi_{\mathrm{eff},ij}-\left(\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\delta_{ij}\right]\frac{dx^j}{dt}+\ldots\end{array} \tag{116}$$

This equation can be written in vector form as

$$\begin{array}{rcl}\frac{d\boldsymbol{v}}{dt} & \cong & -\frac{1}{2}\boldsymbol{\nabla}\left(4\phi_{\mathrm{eff}}-\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)-2\frac{\partial\boldsymbol{A}_{\mathrm{eff}}}{\partial t}+2\boldsymbol{v}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}_{\mathrm{eff}}\right)\\ & & +\frac{1}{2c^2}\frac{\partial}{\partial t}\left(4\phi_{\mathrm{eff}}+\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}:\overline{\overline{\boldsymbol{I}}}\right)\boldsymbol{v}-\frac{2}{c^2}\frac{\partial\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}}{\partial t}\cdot\boldsymbol{v}+\mathcal{O}[v^2].\end{array} \tag{117}$$

Replacing the tensor $\overline{\overline{\boldsymbol{\psi}}}_{\mathrm{eff}}$ by its integrated form (63)

$$\begin{array}{rcl}\frac{d\boldsymbol{v}}{dt} & \cong & -\frac{1}{2}\boldsymbol{\nabla}\phi_{\mathrm{eff}}-2\frac{\partial\boldsymbol{A}_{\mathrm{eff}}}{\partial t}+2\boldsymbol{v}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}_{\mathrm{eff}}\right)+\frac{3}{2c^2}\frac{\partial\phi_{\mathrm{eff}}}{\partial t}\boldsymbol{v}\\ & & +\left(\boldsymbol{\nabla}+\frac{\boldsymbol{v}}{c^2}\frac{\partial}{\partial t}\right)\left(\frac{G}{c^2}\int\frac{\overline{\overline{\boldsymbol{T}}}\left(\boldsymbol{r}',t'\right):\overline{\overline{\boldsymbol{I}}}}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r'\right)\\ & & -\frac{4}{c^2}\frac{\partial}{\partial t}\left(\frac{G}{c^2}\int\frac{\overline{\overline{\boldsymbol{T}}}\left(\boldsymbol{r}',t'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r'\right)\cdot\boldsymbol{v}+\mathcal{O}[v^2]\end{array} \tag{118}$$

This expression describes the motion of a non relativistic test particle inside a fully relativistic gravitationally "polarized" and "magnetized" fluid. Recall that the effective gravitational potentials are given in integral form by equations (62) and (63)

$$\phi_{\mathrm{eff}}\left(\overrightarrow{r},t\right)=-\frac{2G}{c^2}\int\frac{U\left(\boldsymbol{r}',t'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r' \quad \text{and} \quad \boldsymbol{A}_{\mathrm{eff}}\left(\boldsymbol{r},t\right)=\frac{2G}{c^2}\int\frac{\boldsymbol{G}\left(\boldsymbol{r}',t'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r', \tag{119}$$

with $t'=t-\left|\boldsymbol{r}-\boldsymbol{r}'\right|/c$. Neglecting the stress contributions ($\overline{\overline{\boldsymbol{T}}}\sim 0$) and considering only the dominant mass contributions, equation (118) in the quasi-static case ($\partial/\partial t\sim 0$) corresponds to the equation of motion of a test particle in the gravitomagnetic field obtained by Thirring [2][3].

Using the definitions of the effective gravitoelectric and gravitomagnetic fields in terms of the potentials ϕ_{eff} and $\boldsymbol{A}_{\mathrm{eff}}$, and also using the harmonic gauge condition $c^{-2}\partial\phi_{\mathrm{eff}}/\partial t=-\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\mathrm{eff}}$, the previous equation of motion can be written in a form similar to the Lorentz force law, which shows additional terms related to the vector potential and to the stress tensor:

$$\begin{array}{rcl}\frac{d\boldsymbol{v}}{dt} & \cong & \frac{1}{2}\boldsymbol{E}_{\mathrm{eff}}+2\boldsymbol{v}\times\boldsymbol{B}_{\mathrm{eff}}-\frac{3}{2}\left(\frac{\partial\mathbf{A}_{\mathrm{eff}}}{\partial t}+\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\mathrm{eff}}\right)\boldsymbol{v}\right)\\ & & +\left(\boldsymbol{\nabla}+\frac{\boldsymbol{v}}{c^2}\frac{\partial}{\partial t}\right)\left(\frac{G}{c^2}\int\frac{\overline{\overline{\boldsymbol{T}}}\left(\boldsymbol{r}',t'\right):\overline{\overline{\boldsymbol{I}}}}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r'\right)\\ & & -\frac{4}{c^2}\frac{\partial}{\partial t}\left(\frac{G}{c^2}\int\frac{\overline{\overline{\boldsymbol{T}}}\left(\boldsymbol{r}',t'\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r'\right)\cdot\boldsymbol{v}+\mathcal{O}[v^2].\end{array} \tag{120}$$

In vacuum the metric tensor perturbations are given by the definition (84)

$$\begin{cases} \widetilde{h}_{00} &= -\dfrac{2\phi_g}{c^2} \\ \widetilde{h}_{0i} &= \widetilde{h}_{i0} = \dfrac{2A_{g,i}}{c} \\ \widetilde{h}_{ij} &= \widetilde{h}_{ji} = -\dfrac{2\phi_g}{c^2}\left(\delta_{ij} - 2e_ie_j\right) + \dfrac{2}{c^2}\left(\dfrac{a_{\perp,i}e_j + e_ia_{\perp,j}}{2}\right) \\ \widetilde{h} &= \dfrac{2}{c^2}\left(\dfrac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right) : \overline{\overline{\boldsymbol{I}}} = 0 \end{cases} \tag{121}$$

and the linearized Christoffel symbols by

$$\begin{cases} \Gamma^i_{00} &= \dfrac{1}{c^2}\partial_i\phi_g + \dfrac{2}{c^2}\dfrac{\partial A_{g,i}}{\partial t} \\ \Gamma^i_{0j} &= \dfrac{1}{c}\left(\partial_jA_{g,i} - \partial_iA_{g,j}\right) - \dfrac{1}{c^3}\dfrac{\partial}{\partial t}\left[\phi_g\left(\delta_{ji} - 2e_je_i\right) - \left(\dfrac{a_{\perp,j}e_i + e_ja_{\perp,i}}{2}\right)\right] \\ \Gamma^i_{jk} &= -\dfrac{1}{c^2}\partial_j\left[\phi_g\left(\delta_{ki} - 2e_ke_i\right) - \left(\dfrac{a_{\perp,k}e_i + e_ka_{\perp,i}}{2}\right)\right] \\ & \quad -\dfrac{1}{c^2}\partial_k\left[\phi_g\left(\delta_{ij} - 2e_ie_j\right) - \left(\dfrac{a_{\perp,i}e_j + e_ia_{\perp,j}}{2}\right)\right] \\ & \quad +\dfrac{1}{c^2}\partial_i\left[\phi_g\left(\delta_{jk} - 2e_je_k\right) - \left(\dfrac{a_{\perp,j}e_k + e_ja_{\perp,k}}{2}\right)\right] \\ \Gamma^0_{00} &= \dfrac{1}{c^3}\dfrac{\partial\phi_g}{\partial t} \\ \Gamma^0_{0j} &= \dfrac{1}{c^2}\partial_j\phi_g \\ \Gamma^0_{jk} &= -\dfrac{1}{c}\left(\partial_jA_{g,k} + \partial_kA_{g,j}\right) - \dfrac{1}{c^3}\dfrac{\partial}{\partial t}\left[\phi_g\left(\delta_{jk} - 2e_je_k\right) - \left(\dfrac{a_{\perp,j}e_k + e_ja_{\perp,k}}{2}\right)\right] \end{cases} \tag{122}$$

The equation of motion becomes

$$\begin{aligned} \frac{d^2x^i}{dt^2} &= -\partial_i\phi_g - 2\frac{\partial A_{g,i}}{\partial t} - 2\left(\partial_jA_{g,i} - \partial_iA_{g,j}\right)\frac{dx^j}{dt} + \frac{1}{c^2}\frac{\partial\phi_g}{\partial t}\frac{dx^i}{dt} \\ &\quad + \frac{2}{c^2}\left(\partial_j\phi_g\right)\frac{dx^j}{dt}\frac{dx^i}{dt} - \frac{1}{c^2}\left(\partial_jA_{g,k} + \partial_kA_{g,j}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^i}{dt} \\ &\quad + \frac{2}{c^2}\frac{\partial}{\partial t}\left[\phi_g\left(\delta_{ji} - 2e_je_i\right) - \left(\frac{a_{\perp,j}e_i + e_ja_{\perp,i}}{2}\right)\right]\frac{dx^j}{dt} \\ &\quad + \frac{1}{c^2}\partial_j\left[\phi_g\left(\delta_{ki} - 2e_ke_i\right) - \left(\frac{a_{\perp,k}e_i + e_ka_{\perp,i}}{2}\right)\right]\frac{dx^j}{dt}\frac{dx^k}{dt} \\ &\quad + \frac{1}{c^2}\partial_k\left[\phi_g\left(\delta_{ij} - 2e_ie_j\right) - \left(\frac{a_{\perp,i}e_j + e_ia_{\perp,j}}{2}\right)\right]\frac{dx^j}{dt}\frac{dx^k}{dt} \\ &\quad - \frac{1}{c^2}\partial_i\left[\phi_g\left(\delta_{jk} - 2e_je_k\right) - \left(\frac{a_{\perp,j}e_k + e_ja_{\perp,k}}{2}\right)\right]\frac{dx^j}{dt}\frac{dx^k}{dt} \\ &\quad - \frac{1}{c^4}\frac{\partial}{\partial t}\left[\phi_g\left(\delta_{jk} - 2e_je_k\right) - \left(\frac{a_{\perp,j}e_k + e_ja_{\perp,k}}{2}\right)\right]\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^i}{dt}. \end{aligned} \tag{123}$$

This equation can be written in vector form as

$$\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= -\boldsymbol{\nabla}\phi_g - 2\frac{\partial\boldsymbol{A}_g}{\partial t} + 2\boldsymbol{v}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}_g\right) + \frac{1}{c^2}\frac{\partial\phi_g}{\partial t}\boldsymbol{v} \\ &\quad + 2\left(\boldsymbol{\nabla}\phi_g\right)\cdot\frac{\boldsymbol{v}\boldsymbol{v}}{c^2} - 2\frac{\boldsymbol{v}}{c}\cdot\left(\boldsymbol{\nabla}\boldsymbol{A}_g\right)\cdot\frac{\boldsymbol{v}}{c}\boldsymbol{v} \\ &\quad + \frac{2}{c^2}\frac{\partial}{\partial t}\left[\phi_g\left(\overline{\overline{\boldsymbol{I}}} - 2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}\right) - \left(\frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right)\right]\cdot\boldsymbol{v} \\ &\quad + 2\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{\nabla}\left[\phi_g\left(\overline{\overline{\boldsymbol{I}}} - 2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}\right) - \left(\frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right)\right]\cdot\frac{\boldsymbol{v}}{c} \\ &\quad - \boldsymbol{\nabla}\left[\phi_g\left(\overline{\overline{\boldsymbol{I}}} - 2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}\right) - \left(\frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right)\right] : \frac{\boldsymbol{v}\boldsymbol{v}}{c^2} \\ &\quad - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\phi_g\left(\overline{\overline{\boldsymbol{I}}} - 2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}\right) - \left(\frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right)\right] : \frac{\boldsymbol{v}\boldsymbol{v}}{c^2}\boldsymbol{v}. \end{aligned} \tag{124}$$

Assuming non relativistic motion of the test particle

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= -\boldsymbol{\nabla}\phi_g - 2\frac{\partial \boldsymbol{A}_g}{\partial t} + 2\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}_g) + \frac{3}{c^2}\frac{\partial\phi_g}{\partial t}\boldsymbol{v} \\ &\quad -\frac{2}{c^2}\frac{\partial}{\partial t}\underbrace{\left(2\phi_g\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}\right)}_{\overline{\overline{\boldsymbol{\psi}}}_g}\cdot\boldsymbol{v} + \mathcal{O}\left[\beta^2\right].\end{aligned} \tag{125}$$

Using the harmonic gauge conditions

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}_g + \frac{1}{c^2}\frac{\partial\phi_g}{\partial t} = 0 \quad \text{and} \quad \boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\psi}}}_g = \boldsymbol{\nabla}\phi_g + \frac{\partial\boldsymbol{A}_g}{\partial t} = -\boldsymbol{E}_g, \tag{126}$$

the previous equation can be written as

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= -\boldsymbol{\nabla}\phi_g + 2\boldsymbol{v}\times\underbrace{(\boldsymbol{\nabla}\times\boldsymbol{A}_g)}_{\boldsymbol{B}_g} - 3\underbrace{(\boldsymbol{\nabla}\cdot\boldsymbol{A}_g)}_{-c^{-2}\partial\phi_g/\partial t}\boldsymbol{v} \\ &\quad +2\left(\underbrace{\boldsymbol{\nabla}\cdot\left(\phi_g\overline{\overline{\boldsymbol{I}}} - \overline{\overline{\boldsymbol{\psi}}}_g\right)}_{-\partial\boldsymbol{A}_g/\partial t} - \frac{1}{c^2}\frac{\partial\overline{\overline{\boldsymbol{\psi}}}_g}{\partial t}\cdot\boldsymbol{v}\right) + \mathcal{O}\left[\beta^2\right],\end{aligned} \tag{127}$$

where

$$\overline{\overline{\boldsymbol{\psi}}}_g = 2\phi_g\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}} + \frac{\boldsymbol{a}_\perp\widehat{\boldsymbol{e}} + \widehat{\boldsymbol{e}}\boldsymbol{a}_\perp}{2}. \tag{128}$$

In the quasi-static case the equation of motion of the test particle becomes simply

$$\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}\phi_g + 2\boldsymbol{v}\times\boldsymbol{B}_g + \mathcal{O}\left[\beta^2\right]. \tag{129}$$

The dynamics of the fluid source is ruled by a free-boundary fluid-vacuum interface with emission (or absorption) of gravitational energy. In the far-field region the test particle is subjected to a nearly plane gravitational wave with fixed orientation $\widehat{\boldsymbol{e}}$. Hence

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &\cong \boldsymbol{\nabla}\phi_g + 2\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A}_g) - 4\left(\widehat{\boldsymbol{e}}\cdot\boldsymbol{\nabla}\phi_g\right)\widehat{\boldsymbol{e}} \\ &\quad -\left(\boldsymbol{\nabla}\cdot\boldsymbol{a}_\perp\right)\widehat{\boldsymbol{e}} - \widehat{\boldsymbol{e}}\cdot\boldsymbol{\nabla}\boldsymbol{a}_\perp - \frac{\widehat{\boldsymbol{e}}\cdot\boldsymbol{v}}{c^2}\frac{\partial\boldsymbol{a}_\perp}{\partial t} - \frac{\widehat{\boldsymbol{e}}}{c^2}\boldsymbol{v}\cdot\frac{\partial\boldsymbol{a}_\perp}{\partial t} + \mathcal{O}\left[\beta^2\right].\end{aligned} \tag{130}$$

Taking the scalar product with $\boldsymbol{v}$

$$\frac{1}{2}\frac{dv^2}{dt} \cong \boldsymbol{v}\cdot\left(\boldsymbol{\nabla}\phi_g - \widehat{\boldsymbol{e}}\cdot\boldsymbol{\nabla}\boldsymbol{a}_\perp\right) - \left(\widehat{\boldsymbol{e}}\cdot\boldsymbol{v}\right)\left(4\widehat{\boldsymbol{e}}\cdot\boldsymbol{\nabla}\phi_g + \boldsymbol{\nabla}\cdot\boldsymbol{a}_\perp + \frac{2}{c^2}\boldsymbol{v}\cdot\frac{\partial\boldsymbol{a}_\perp}{\partial t}\right) + \mathcal{O}\left[\beta^2\right]. \tag{131}$$

Note that, as expected, the gravitomagnetic field $\boldsymbol{B}_g$ does not work directly, but the gravitoelectric field generated by the changing gravitomagnetic field (the gravitoelectromotive force $-\partial\boldsymbol{A}_g/\partial t$) can do work on the test particle. Considering the Newtonian limit in the above equation, one verifies that the velocity $\boldsymbol{v}$ must be perpendicular to the direction of propagation $\widehat{\boldsymbol{e}}$ of the gravitational wave, that is, the condition $\widehat{\boldsymbol{e}}\cdot\boldsymbol{v} = 0$ gives the correct infinite-distance limit

$$\frac{1}{2}\frac{dv^2}{dt} \cong \boldsymbol{v}\cdot\left(\boldsymbol{\nabla}\phi_g - \widehat{\boldsymbol{e}}\cdot\boldsymbol{\nabla}\boldsymbol{a}_\perp\right) \text{ , where } \quad \widehat{\boldsymbol{e}}\cdot\boldsymbol{v} = 0. \tag{132}$$

The $\widehat{\boldsymbol{e}}\cdot\boldsymbol{v} = 0$ condition corresponds to the last two gauge conditions that may be imposed in the far-field region. This is also consistent with the generation of the vector potential $\boldsymbol{A}$ by the transverse matter currents in the source.

Alternatively, neglecting the interaction with the gravitational wave, i.e., taking $\boldsymbol{a}_\perp \cong 0$ with $\partial\phi_g/\partial t \cong 0$ and $\partial\boldsymbol{A}_g/\partial t \cong 0$ (non-radiating or low frequency regime) but keeping the relativistic corrections for the test particle, the general equation of motion in vacuum gives (note that $\left(2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}-\overline{\overline{\boldsymbol{I}}}\right)\cdot\boldsymbol{v}$ rotates the vector $\boldsymbol{v}$ through 180° about the direction $\widehat{\boldsymbol{e}}$)

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= -\boldsymbol{\nabla}\phi_g-\boldsymbol{\nabla}\left[\phi_g\left(\overline{\overline{\boldsymbol{I}}}-2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}\right)\right]:\frac{\boldsymbol{v}\boldsymbol{v}}{c^2}\\ &\quad+2\frac{\boldsymbol{v}}{c}\cdot\left(\boldsymbol{\nabla}\phi_g\right)\frac{\boldsymbol{v}}{c}+2\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{\nabla}\left[\phi_g\left(\overline{\overline{\boldsymbol{I}}}-2\widehat{\boldsymbol{e}}\widehat{\boldsymbol{e}}\right)\right]\cdot\frac{\boldsymbol{v}}{c}\\ &\quad+2\boldsymbol{v}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}_g\right)-2\boldsymbol{v}\frac{\boldsymbol{v}}{c}\cdot\left(\boldsymbol{\nabla}\boldsymbol{A}_g\right)\cdot\frac{\boldsymbol{v}}{c}.\end{aligned} \tag{133}$$

Here $\widehat{\boldsymbol{e}}$ corresponds to the reference direction of the intrinsic gravitational wave with negligible amplitude outside the sources. For vanishing gravitational waves this direction can still be affected by refraction effects in the near gravitoelectromagnetic field. Neglecting these effects and using $\widehat{\boldsymbol{e}}\cdot\boldsymbol{v}=0$ gives

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= -\left(1+\frac{v^2}{c^2}\right)\boldsymbol{\nabla}\phi_g+4\frac{\boldsymbol{v}}{c}\cdot\left(\boldsymbol{\nabla}\phi_g\right)\frac{\boldsymbol{v}}{c}\\ &\quad+2\boldsymbol{v}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}_g\right)-2\boldsymbol{v}\frac{\boldsymbol{v}}{c}\cdot\left(\boldsymbol{\nabla}\boldsymbol{A}_g\right)\cdot\frac{\boldsymbol{v}}{c}.\end{aligned} \tag{134}$$

This equation describes the perihelion precession rate of planetary orbits due to relativistic corrections of the test particle motion, a topic that will be treated in the part III article.

V. COMMENTS AND CONCLUSIONS

A consistent set of hydrodynamic and Maxwell equations for the gravitoelectromagnetic field was obtained, in the first article of a three parts work, applying Hamilton's principle to a fully-relativistic perfect fluid, leading to an extended gravitoelectromagnetic model [1]. Then, in the present second part article, the total energy-momentum tensor given by the sum of the fluid and field tensors is used to construct a novel form for the metric tensor perturbations, both inside the fluid and in the external (vacuum) region. The perfect fluid energy-momentum tensor is modified by the addition of the free-field tensor, in an approach similar to the Landau-Lifshitz pseudotensor proposition. The combined fluid-field tensor correctly describes the fluid dynamics in flat space, satisfies the energy-momentum conservation equation, and can be used to determine the metric perturbations according to linearized gravity. Since the full gravitoelectromagnetic field tensor has vanishing trace it does not contribute directly to the time-space curvature, but it affects the dynamics of the curvature generating matter. Besides the time-time and mixed time-space components, usually associated with the gravitoelectromagnetic field, the metric perturbations include spatial components related to the fluid stresses. These stresses act in the formation of gravitational waves, on a higher order in the relativistic fluid velocity. It is argued that the energy carried by the fluid oscillations (gravity waves) and by the internal gravitoelectromagnetic waves modify the metric, forming gravitational waves that may propagate into vacuum. In vacuum, the gravitoelectromagnetic field associated with the internal waves may affect the propagation of the gravitational waves near the fluid-vacuum interface. Further out, in the far-field region, the gravitational waves decouple from the gravitoelectromagnetic waves, transmitting net energy away from the source. The derived geodesic equation can be used to describe the motion of a test particle both inside the fluid and in the external region. Application of the geodesic equation is deferred in this series to the third part article entitled "Extended gravitoelectromagnetism. III. Mercury's perihelion precession".

In retrospect, the article develops a theory of gravitoelectromagnetism which is compatible with the formation of gravitational waves, that is, with a full set of field equations for the gravitoelectromagnetic fields interacting with a fluid source. It must be pointed out that Faraday's law for the fields follows naturally from the present formulation. The metric perturbation includes time-time, mixed time-space and, in particular, purely spatial tensor components which describe the formation of gravitational waves. The gravitoelectromagnetic force represented by the time rate of change of the vector potential is responsible for driving the waves, providing the connection with the transverse mass current fluctuations. An essential part of the theory is that only two gauge conditions are imposed on the metric tensor perturbation in the vacuum region. These restricted set of conditions defines the near-field region, close to the fluid-vacuum interface. The full vacuum gauge conditions are approximately satisfied by the gravitational waves in the far-field region only. In fact, the imposition of four gauge conditions on the metric tensor perturbations in vacuum breaks the connection between the gravitoelectromagnetic force and the gravitational waves, restricting the theory to either quasi-static fields or pure gravitational waves in vacuum.

As a final comment, recall that the gravitoelectromagnetic waves considered in the part I article of the series can also propagate away from the source into the radiation zone. The pure gravitoelectromagnetic waves are based on a vector field and have different propagating characteristics, polarizations and radiation patterns compared to the gravitational waves associated with the tensorial field of metric perturbations [9]. Arguably, the pure gravitoelectromagnetic waves correspond to a weakly relativistic stage of the gravitational waves (of the order of β in the source oscillations), and the tensor related gravitational waves to a strong relativistic stage (of the order of β^2). This is a topic that requires further investigation.

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