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Citation: Physics of Plasmas 25, 102308 (2018); doi: 10.1063/1.5049237
View online: https://doi.org/10.1063/1.5049237
View Table of Contents: http://aip.scitation.org/toc/php/25/10
Published by the American Institute of Physics

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D. S. Oliveira[1] and R. M. O. Galvão[1,2]
[1]Institute of Physics, University of São Paulo, 05508-090 São Paulo, Brazil
[2]Instituto Nacional de Pesquisas Espaciais, 12227-010 São José dos Campos, Brazil

(Received 20 July 2018; accepted 3 October 2018; published online 24 October 2018)

Non-Maxwellian distribution functions are frequently observed in space and laboratory plasmas in (quasi-) stationary states, usually resulting from long-range nonlinear wave-particle interactions [P. H. Yoon, Phys. Plasmas **19**, 012304 (2012)]. Since the collisional transport described by the Boltzmann equation with the standard collisional operator implies that the plasma distribution function evolves inexorably towards a Maxwellian, the description of the transport for stationary states outside of equilibrium requires a different formulation. In this work, we approach this problem through the non-extensive statistics formalism based on the Tsallis entropy. The basic framework of the kinetic model and the required generalized form of the collision operator are self-consistently derived. The fluid equations and the relevant transport coefficients for electrons are then found employing the method of Braginskii. As an illustrative application of the model, we employ this formalism to analyze the heat flux in solar winds. *Published by AIP Publishing.*
https://doi.org/10.1063/1.5049237

I. INTRODUCTION

A common distinctive feature of most laboratory and space plasmas is that of being far from thermodynamic equilibrium. In the turbulent state usually observed, in particular when the underlining plasma modes have long-range correlations and/or relevant wave-particle interactions, the particle velocity distribution functions are quite far from a Maxwellian, with long energetic tails, especially for electrons, but also for another species in some cases.[1,2] One of the first observations of long-tail electron distribution functions was made by Vasyliunas, when analyzing low-energy electron fluxes in the magnetosphere measured by the OGO1 and OGO3 satellites.[3] In order to explain the data, an empirical distribution function, approaching a power law at high energies, was introduced, which became known as the κ-distribution function. Recent data from STEREO (Solar Terrestrial Probes Program) show clearly long tails in the quite time superhalo electron velocity distribution function, which are well modelled by the κ-distribution function.[4,5] Due to their wide range of applications, κ-distribution functions have been employed in many plasma studies over the years, in particular regarding the modification of the dispersion relation of different kinetic plasma modes and the evolution of nonlinear instabilities.[6,7]

In tokamak plasmas, long-tail electron distributions have been observed over the whole plasma column (edge, confinement region, and core) and related to different mechanisms and instabilities, such as magnetic reconnection, high-energy ions, non-local electron transport, neutral ionization, internal kink modes, sawtooth instabilities, electron fishbones, and plasma heating.[8–11] In particular, bi-Maxwellian distribution functions, an empirical model for long tail distributions used in the same context of the κ-distributions in space plasmas, were fitted to the experimental data in the plasma edge during Ion Cyclotron Resonance Heating (ICRH) and Neutral Beam Injection (NBI) experiments.[12,13]

In order to introduce non-thermal effects in the Boltzmann-based models, alternative fluid equations approximating different long-tail distribution functions by a series of Maxwellians, with the coefficients of proportionality determined by numerically fitting the experimental data, have been lately considered, especially in numerical simulations.[2] Despite their success in recovering some quantitative experimental results, these models lack a convincing theoretical basis, in particular, regarding the adoption of the collision operator from the Boltzmann statistics,[14] which implies that the plasma has to evolve towards thermodynamic equilibrium described by a Maxwellian. Naturally, this is not consistent with stationary non-Maxwellian distribution functions since they are not stationary maxima of the Boltzmann entropy.

An entirely different theoretical framework to model systems outside thermodynamic equilibrium was pioneered by Tsallis.[15,16] In his approach, a generalized form of the Boltzmann entropy is introduced

$$S_q = k_B \frac{1 - \sum_\mu p_\mu^q}{q-1}, \tag{1}$$

where k_B is the Boltzmann constant, p_μ is the normalized probability of the state μ, and, for $q \to 1$, the above expression recovers the Boltzmann entropy (S_B).The parameter q is interpreted as modelling the effect of strong dynamical correlations within the system, which substantially modifies the nature of the phase space occupation.[17] In plasma physics, such interpretation is consistent with the generation of the (quasi-) stationary power-law equilibrium distribution functions by Langmuir turbulence and kinetic Alfvén and low-hybrid wave-particle interactions.[18,19] Since the value of q can somewhat be adjusted to model different types of long-range correlations,[16] many other situations in plasmas can be described by the Tsallis formalism, as indeed it has been reported in the literature; for instance, the equilibrium density after turbulent relaxation in a pure electron plasma,[20] superdiffusion transport in dusty plasmas,[21] plasma oscillations,[22] κ-distribution functions as a family of distributions of the Tsallis theory ($\kappa = 1/(q-1)$),[23,24] transport coefficients in the BGK approximation,[25,26] vorticity distribution at the plasma edge of tokamaks,[27] etc.

From a theoretical point of view, the S_q is one of the most robust generalizations of the S_B, sharing its most relevant properties; the main difference between them being the non-additivity of the former.[16] For some time, this feature led to the misconception of the Tsallis entropy being a "non-extensive entropy." However, additivity is a sufficient condition rather than a necessary one; therefore, S_q is subject to the extensiveness in the same way as S_B.[28] Nevertheless, we will keep the "non-extensive entropy" (or non-extensive statistics/theory) terminology that became widespread in the literature.

In this paper, starting just from the equivalent definition of S_q for continuous systems, the closed electron fluid equations, in the limit of weak interactions, are derived from a self-consistent non-extensive kinetic theory (q-kinetic theory). We restrict the analysis to the electron fluid equations in plasmas with only one ionic component, for the sake of simplicity. Therefore, this work is a first step in the development of a plasma transport model based on the Tsallis entropy, which will be extended to include the ion fluid equations. In Sec. II, the continuous formulation of the q-kinetic theory as well as the proper definition of temperature within this framework is discussed. The collisional operator suitable for non-Maxwellian distributions is found in Sec. III with the help of the Kinetic Interaction Principle (KIP),[29] whereas the general aspects of the kinetic model and the Chapman-Enskog (CE) method[30–32] are presented in Secs. IV and V, respectively. Section VI is dedicated to the numerical evaluation of the main transport coefficients. An application in space plasmas is presented in Sec. VII.

II. NON-MAXWELLIAN DISTRIBUTION FUNCTION

The characteristic asymptotic stationary power-law distributions resulting from the wave-particle interactions are well represented by the non-extensive distribution function resulting from the maximization of S_q, as already verified in the literature.[23] Departing from this point, we obtain the distribution function in this section by the maximization of the entropy. This method allows us to define the temperature in order to retain its conventional meaning in the transport approach.

In the continuous formulation, the generalized entropy is defined by[16]

$$S_q \equiv \int d\mathbf{v}\, p \ln_q\left(\frac{1}{p}\right), \tag{2}$$

where $k_B = 1$ (temperature measured in energy units), q is a real number, p is the distribution function, $ln_q(x) = (x^{1-q} - 1)/(1-q)$ is the q-logarithm, and the integrals extend all over the velocity phase space. The normalization conditions, which define the density of particles (n) and the q-mean, are, respectively, given by

$$n = \int d\mathbf{v} p, \tag{3}$$

$$\frac{O_q}{n} = \frac{\int d\mathbf{v}\, O(\mathbf{v}) p^q}{\int d\mathbf{v} p^q}, \tag{4}$$

where O_q is the mean value corresponding to the operator $O(\mathbf{v})$ in velocity space. In Eq. (4), n appears in the left side due the normalization condition and to recover the usual mean when $q \to 1$.

In this approach, the internal energy of the plasma particles becomes

$$\frac{u_q}{n} = \frac{\int d\mathbf{v}\left(\frac{mv^2}{2} + e_a\phi\right)p^q}{\int d\mathbf{v}\, p^q}, \tag{5}$$

where e_a is the electric charge of the particle species, m is the mass, ϕ is the electric potential, and the index "a," which distinguishes electrons and ions, has been suppressed in quantities but the charge, since the calculations in this section are identical for all species. It is important to notice that Eq. (2) is defined up to a proportionality constant in p. However, during the standard variational extremization procedure, which results in the stationary distribution function as presented next, such a constant can be coupled in the Lagrange multipliers and, therefore, disappears in the final expression.

From the standard variational extremization procedure of the Lagrangian of the entropy,[33–35] with the constrains given by Eqs. (3) and (5), the equilibrium distribution function is obtained as

$$p_0 = \beta_n\left[1 - (1-q)\beta_u\left(\frac{mv^2}{2} + e_a\phi - \frac{u_q}{n}\right)\right]^{\frac{1}{1-q}}, \tag{6}$$

where β_n and β_u account for the Lagrange multipliers of the normalization constant (density) and internal energy (temperature dependent), respectively. The above distribution function is the well-known power-law equilibrium distribution function of the non-extensive statistics, which replaces the ordinary Maxwellian distribution in the traditional approach. Henceforward, we limit our analysis to $q > 1$, where long-tail distributions are found (actually, for $q < 1$, p has an upper limit in velocity space given by the Tsallis cut-off, which limits the distribution function[16]).

It is also convenient to rewrite p in terms of the escort distribution functions $f = np^q/(\int d\mathbf{v}\, p^q)$ (q-escort distribution).[16,36] The main advantage of this approach is to transform the q-mean [see Eq. (4)] into the ordinary statistical average as in the Maxwell-Boltzmann statistics. In this new formulation, p_0 in Eq. (6) is rewritten as

$$f_0 = n_0\left(\frac{m\beta_q}{2}\right)^{\frac{3}{2}}\left[1 - (1-q)e_a\beta_q\phi\right]A_q \times \left[1 - (1-q)\beta_q\left(\frac{mv^2}{2} + e_a\phi\right)\right]^{\frac{q}{1-q}}, \tag{7}$$

where the normalization constant A_q, obtained from Eq. (3), is

$$A_q = \begin{cases} \pi^{-\frac{3}{2}}, & q = 1; \\ \dfrac{(q-1)^{\frac{1}{2}}}{\pi^{\frac{3}{2}}} \dfrac{\Gamma\left(\dfrac{1}{q-1}\right)}{\Gamma\left(-\dfrac{1}{2}+\dfrac{1}{q-1}\right)}, & 1 < q < 3, \end{cases} \quad (8)$$

and we also have used $n(\phi)$ and u_q obtained from Eqs. (3) and (5) with the substitution of Eq. (7)

$$\frac{n}{n(\phi=0)} = \frac{n}{n_0} = \left[1-(1-q)e_a\beta_q\phi\right]^{\frac{3}{2}+\frac{1}{1-q}}, \quad (9)$$

$$1-(1-q)e_a\beta_q\phi > 0; \quad u_q = \frac{2}{5-3q}\frac{n}{\beta_q}\left(\frac{3}{2}+e_a\beta_q\phi\right);$$
$$1 < q < \frac{5}{3}, \quad (10)$$

where the upper limits on q correspond to the maximum values beyond which the integrals of f_0 diverge, and β_q is given by

$$\beta_q = \frac{\beta_u n\left[\int d\mathbf{v}\, p^q\right]^{-1}}{1+(1-q)u_q\beta_u\left[\int d\mathbf{v}\, p^q\right]^{-1}}. \quad (11)$$

The definition of temperature in the non-extensive statistics is not unique; the kinetic and the equilibrium temperatures are essentially different from the Lagrangian temperature; they even may have different physical interpretations.[37–40] In the current model, the temperature is defined by the generalized zeroth law[40,41]

$$\left(\frac{\partial S_q}{\partial u_q}\right)_n \left[1+(1-q)S_q/n\right]^{-1} = \frac{1}{T}, \quad (12)$$

where T is the equilibrium temperature of the system measured by a thermometer. By substituting Eqs. (2), (11), and (12) into Eq. (10), we recover the classical internal energy $u_q = \frac{3}{2}nT + ne_a\phi$, where T is understood as the usual average kinetic energy. Hence, the second moment of the kinetic equation, that is, the energy balance equation in Sec. IV B, holds its usual meaning. From Eqs. (2), (11), and (12), we also define the auxiliary temperature

$$\frac{1}{\beta_q} = T_q = \frac{5-3q}{2}T + (1-q)e_a\phi. \quad (13)$$

If the density given in Eq. (9) is expanded at $q \to 1$ and the weak interaction condition (namely, $e_a\phi/T \ll 1$) is applied, i.e., only collisional transport, then

$$n = n_0 e^{-\frac{e_a\phi}{T}}\left(1 - \frac{1-q}{2}\left(\frac{e_a\phi}{T}\right)^2 + \cdots\right) \approx n_0 e^{-\frac{e_a\phi}{T}}, \quad (14)$$

recovering the ordinary expression of the density from Boltzmann statistics. In turn, the Debye length and, therefore, the upper cut-off of the collision cross-section do not change.[42,43] An analogous expansion of the distribution function in Eq. (7) around $q \to 1$ yields Maxwellians multiplied by appropriate coefficients, suggesting that the numerical expansions aforementioned in Sec. I may asymptotically approach the non-extensive distribution functions.

The weak interaction condition is rigorously verified for $q \in [1, 1.4]$, where $2(q-1)/(5-3q) \leq 1$ guarantees that the second term in Eq. (13) is always smaller than the first. This restriction is needed because when $q \to 5/3$, the dependence on T in Eq. (7) is negligible and, therefore, the width of the distribution is set only by ϕ. Finally, applying $e_a\phi/T \ll 1$ and using the self-referential property of the escort distributions,[16] the expressions for the distribution function [Eq. (7)] and T_q [Eq. (13)] yield

$$f_0 \approx n\left(\frac{m}{2T_q}\right)^{\frac{3}{2}} A_q\left[1-(1-q)\frac{mv^2}{2T_q}\right]^{\frac{q}{1-q}}; \quad T_q \approx \frac{5-3q}{2}T. \quad (15)$$

These approximations are compatible with the stationary power-law distribution functions obtained in wave-particle numerical simulations of the observed superhalo electron velocity distribution function[18] and with the electron temperature measurements in solar winds.[44]

III. COLLISIONS IN NON-MAXWELLIAN PLASMAS

Stationary non-Maxwellian distribution functions in plasmas are a consequence of dynamical equilibration rather than collisional relaxation. In fact, the relaxation due collisions drive the system toward the thermodynamic equilibrium, which is described by a Maxwellian. Therefore, since collisions are the main transport mechanism of the fluxes caused by small perturbations of the equilibrium parameters, the kinetic equation based on the Boltzmann statistics has to be modified to include the incompleteness of the collisional relaxation in the non-Maxwellian picture.

Taking advantage of non-Maxwellians being maxima of S_q, an alternative kinetic equation can be derived self-consistently in non-extensive statistics so that f_0 given in Eq. (15) represents a sort of "collisional equilibrium." The main advantage of this approach is to model the transport phenomena of non-Maxwellian stationary plasmas without using the complete kinetic equation with the wave-interaction terms (the long-range correlation terms). Of course, the inclusion of the appropriated modifications in the ordinary kinetic equation, in principle, has to produce the same results for the fluxes obtained in our formulation. However, to verify whether strong dynamical correlations are a sufficient or necessary condition to change the distribution function in phase space is a somewhat difficult task. In fact, this is unnecessary in our approach, since we assume that the non-Maxwellian distribution function is already formed.

It is important to mention that there are other models in the literature that explore the incompleteness of the statistical independence due to long-range correlations leading to extensions of the Boltzmann equation.[45–47] However, none of them allows for a consistent derivation of the fluid equations from the kinetic equation and the correct expression for the collisional operator simultaneously in non-thermal plasmas. For this reason, we employ the Kinetic Interaction Principle (KIP) method, introduced in Ref. 48, to derive the collision operator with the kinetic equation suitable to obtain the fluid equations. The first generalization of the Landau operator was presented for normal q-distributions [Eq. (6)] in Ref. 46. Here, the generalized kinetic equation is derived for q-escort distributions, since in this formulation the determination of the fluid equations from the kinetic theory follows the standard kinetic moments procedure, because of the already mentioned recovery of the standard statistical average.

The KIP method states that the evolution of the distribution function in phase space is set by[29,48]

$$\frac{df}{dt} = \int d\mathbf{v}' d\mathbf{v}_1 d\mathbf{v}_1' \left[\Pi\left(\mathbf{r}, \mathbf{v}' \to \mathbf{v}, \mathbf{v}_1' \to \mathbf{v}_1, t\right) - \Pi\left(\mathbf{r}, \mathbf{v} \to \mathbf{v}', \mathbf{v}_1 \to \mathbf{v}_1', t\right)\right], \tag{16}$$

where $\mathbf{r}$ is the position where collision occurs, *d/dt* is the convective derivative, $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{v}_1$, and $\mathbf{v}_1'$ are, respectively, the incident and target velocity of the particles before and after the collision, and Π is the probability of transitions. Π is further decomposed as a generic combination of positive definite functions

$$\Pi = \mathrm{Tr}\left(\mathbf{r}, \mathbf{v}', \mathbf{v}, \mathbf{v}_1', \mathbf{v}_1, t\right)\gamma\left(f, f'\right)\gamma\left(f', f\right), \tag{17}$$

where Tr is the transition rate and $\gamma(f,f') = a(f)b(f')c(f,f')$, with a, b, and c being positive functions, in which $c(f',f) = c(f,f')$ accounts for the influence of the populations on the collision process. In explicit terms of the a, b, and c functions, Eq. (16) is

$$\frac{df}{dt} = \int d\mathbf{v}' d\mathbf{v}_1 d\mathbf{v}_1 \,\mathrm{Tr}(r, \mathbf{v}', \mathbf{v}, \mathbf{v}_1{}', \mathbf{v}_1, t) \times cc_1\left[a' b a_1{}' b_1 - a b' a_1 b_1{}'\right], \tag{18}$$

where $a = a(f)$, $a' = a(f')$, and so on.

In the weak interaction condition, the collisions are binary and cause only small changes in the particle velocities $|\mathbf{\Delta}| = |\mathbf{v} - \mathbf{v}_1| \ll (|\mathbf{v}|, |\mathbf{v}_1|)$, and functions a and b in Eq. (17) can be expanded as power series. The general steps of the calculations can be found in Ref. 46; the resulting kinetic equation is

$$\frac{df}{dt} = \frac{\partial}{\partial v_\mu}\int d\mathbf{v}_1 K_{\mu\nu}\frac{cc_1}{m}\left[\frac{g_1 h}{m}\frac{\partial f}{\partial v_\mu} - \frac{g h_1}{m_1}\frac{\partial f_1}{\partial v_{1\mu}}\right], \tag{19}$$

where

$$g = ab; \quad h = \frac{da}{df}b - a\frac{db}{df}; \quad K_{\mu\nu} = \int d\mathbf{v}_1 \mathrm{Tr}(\mathbf{r}, \mathbf{v}, \mathbf{v}_1, t; \mathbf{\Delta}). \tag{20}$$

The relations for g_1 and h_1 are analogous and the indices μ and ν stand for the Cartesian coordinates x, y, and z. The tensor $K_{\mu\nu}$ depends on the cross section of the collisions and it is calculated from the standard Newton mechanics[42] as

$$K_{\mu\nu} = \frac{2\pi e^2 e_1^2 \lambda}{m} U_{\mu\nu} = \frac{2\pi e^2 e_1^2 \lambda}{m}\frac{\delta_{\mu\nu}u^2 - u_\mu u_\nu}{u^3}, \tag{21}$$

where m is the incoming particle mass, $u_\mu = v_\mu - v_{1\mu}$ is the relative velocity, e and e_1 are the charges of the particles involved in the binary collision, and $\lambda = \ln(\lambda_D / r_{imp})$ is the Coulomb logarithm with λ_D the Debye length and r_{imp}, the impact parameter.

The functions h are obtained from $d^2G/df^2 = h/g$, where $S_q = \int d\mathbf{v} G(f)$.[29,48] In order to define all functions uniquely, the c and g functions have to be chosen properly. Since the collisions are binary and the Coulomb force is symmetric, the instantaneous process is independent of the particle populations and, therefore, $cc_1 = 1$. In the weak interaction limit, the integrals in Eq. (19) must approach a diffusive process in phase space.[49] Accordingly, the choice $g = f$ ($g_1 = f_1$) enables interpreting these integrals as the h (h_1) weighted average of the momentum transfer from f to f_1 (or f_1 to f).

The final expression of the collisional operator in our model is

$$C(f, f_1) = \frac{2\pi e^2 e_1^2 \lambda}{m}\frac{\partial}{\partial v_\nu}\int d\mathbf{v}_1 U_{\mu\nu}\left[\frac{f_1}{m}\frac{\partial f^*}{\partial v_\mu} - \frac{f}{m_1}\frac{\partial f_1^*}{\partial v_{1\mu}}\right], \tag{22}$$

where f^* (or f_1^*) is

$$f^* = \frac{f^{\frac{1}{q}}}{q k_q} \approx \frac{5 - 3q}{2} n\left(\frac{m}{2T_q}\right)^{\frac{3}{2}} A_q \left[\frac{f}{n\left(\frac{m}{2T_q}\right)^{\frac{3}{2}} A_q}\right]^{\frac{1}{q}}, \tag{23}$$

and we have used, in advance, that the solutions of interest are $f = f_0 + \delta f$, where δf is the first order solution and $\delta f / f_0 \ll 1$, which allows $k_q(f) \approx k_q(f_0) = \int d\mathbf{v} f_0^{1/q}/n$.

In Eq. (22), the parameter q does not appear explicitly and no further hypothesis besides the particles being charged were made. Therefore, the extension of the results so far obtained to all particle species in the plasma, whose populations are described by different distribution functions, with different parameters or q's, is straightforward

$$C_a = \sum_b \frac{2\pi e_a^2 e_b^2 \lambda_{ab}}{m_a} \times \frac{\partial}{\partial v_{a\nu}}\int d\mathbf{v}_b U_{\mu\nu}\left[\frac{f_b}{m_a}\frac{\partial f_a^*}{\partial v_{a\mu}} - \frac{f_a}{m_b}\frac{\partial f_b^*}{\partial v_{b\mu}}\right], \tag{24}$$

and the non-extensive multicomponent plasma kinetic equation in terms of q-escort distributions is

$$\frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \nabla f_a + \frac{\mathbf{F}_a}{m_a} \cdot \nabla_{\mathbf{v}_a} f_a = C_a, \tag{25}$$

where $\mathbf{v}_a$ is the velocity of the "a" species and $\mathbf{F}_a$ is the external force acting on these, and $q \to 1$ recovers the Landau-Boltzmann equations.

The proof of the constraint relations (conservation of mass, momentum, and energy) as well as the H-theorem is given in Refs. 29, 46, and 48 in general terms, i.e., before the supposition of a specific statistics, including Tsallis statistics. Further properties of the q-Landau operator are shown in Appendix A.

IV. KINETIC MODEL

A. Electron-ion approximations

Due to the mass disparity between electrons and ions, $m/m_i \ll 1$, the velocity of the ions is, in general, much smaller than that of the electrons. This condition enables the expansion of $U_{\mu\nu}$ in the power series of the ion velocity

$$U_{\mu\nu} = V_{\mu\nu} - \frac{\partial V_{\mu\nu}}{\partial v_\xi} v_{i\xi} + \frac{1}{2}\frac{\partial^2 V_{\mu\nu}}{\partial v_\eta \partial v_\xi} v_{i\xi} v_{i\eta}, \tag{26}$$

where $V_{\mu\nu} = U_{\mu\nu}(\mathbf{v}_i = 0)$ and we took, for convenience, the coordinate system where $\mathbf{V}_i = 0$. Then, the above expression can be substituted into Eq. (24) and integrated over $\mathbf{v}_i$, with the boundary condition $f(v \to \infty) = 0$, yielding the approximative expression of the electron-ion collision operator

$$C_{ei} = \frac{2\pi e^2 e_i^2 n_i \lambda}{m}\frac{\partial}{\partial v_\nu}\left[V_{\mu\nu}\frac{\partial f^*}{\partial v_\mu} - \frac{m}{m_i}\left(2\frac{v_\mu}{v^3}\frac{n_i^*}{n_i} f + \frac{3v_\mu v_\nu - v^2\delta_{\mu\nu}}{v^5}\frac{T_i}{m}\frac{\partial f^*}{\partial v_\mu}\right)\right], \tag{27}$$

where $n_i^* = \int d\mathbf{v}_i f_i^*$. Neglecting terms of $\mathcal{O}(m/m_i)$, the principal part of the C_{ei} is

$$C'_{ei}(f) = \frac{2\pi e^2 e_i^2 n \lambda}{m}\frac{\partial}{\partial v_\nu}\left[V_{\mu\nu}\frac{\partial f^*}{\partial v_\mu}\right], \tag{28}$$

where the local neutrality $n \approx n_i$ was invoked and, except for f^*, the above expression is equal to the classical operator.[31,50] Furthermore, it can be also verified that the same expression for the e-i collision frequency for the zeroth order collision classical operator is held,[43] namely,

$$\omega_{ei} = \frac{3\sqrt{\pi}}{4\tau}\left(\frac{v_T}{v}\right)^3, \tag{29}$$

where $v_T^2 = 2T/m$ is the thermal velocity and τ is the relaxation time defined as

$$\tau = \frac{3\sqrt{m}T^{\frac{3}{2}}}{4\sqrt{2\pi}e^2 e_i^2 n\lambda}. \tag{30}$$

B. Transport equations

From Eq. (25), it follows that, for a fully ionized single species plasma, in the presence of stationary electromagnetic fields, the kinetic equation in terms of the peculiar velocity of the electrons, $\mathbf{v} = \mathbf{v}' - \mathbf{V}$ (the velocity of the electrons is now $\mathbf{v}'$), is

$$\frac{df}{dt} + \mathbf{v}\cdot\nabla f + \left(\frac{e}{m}\left(\mathbf{E}' + \frac{\mathbf{v}}{c}\times B\right) - \frac{d\mathbf{V}}{dt}\right)\cdot\nabla_v f = C_e, \tag{31}$$

where f is the electron distribution function, $\mathbf{E}' = \mathbf{E} + \mathbf{V} \times \mathbf{B}/c$, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields in the laboratory frame, respectively, and $C_e = C_{ee}(f) + C_{ei}(f)$ is the total collisional operator accounting for electron-electron collisions (e-e collisions) and electron-ion collisions (e-i collisions). Since the e-i collision operator in Eq. (28) is independent of the distribution function of the ions, the evolution of the electron distribution function is obtained independently of the evolution of f_i as well as its fluid equations.

In the weak interaction limit, only the first three moments of the kinetic equation are enough for a reasonable approximation of the fluid equations.[32] Since the q-escort approach holds the ordinary statistical average and Eq. (31) has the exact form of the kinetic equation for the Maxwell-Boltzmann statistics,[43,50] the first three moments (namely, multiplying the kinetic equation by either $1, m\mathbf{v}, mv^2/2$ and integrating over $\mathbf{v}$) recover the classical transport equation system

$$\frac{dn}{dt} + n\nabla\cdot\mathbf{V} = 0, \tag{32}$$

$$nm\frac{d\mathbf{V}}{dt} + \nabla p = en\mathbf{E}' + \mathbf{R}, \tag{33}$$

$$\frac{3}{2}n\frac{dT}{dt} + p\nabla\cdot\mathbf{V} + \nabla\cdot\mathbf{q} = Q, \tag{34}$$

where the following quantities have been introduced:

$$p = \int d\mathbf{v}'\frac{mv'^2}{2}f = nT; \quad Q = \int d\mathbf{v}'\frac{mv'^2}{2}C_e; \quad \mathbf{q} = \int d\mathbf{v}'\frac{mv'^2}{2}\mathbf{v}'f; \quad \mathbf{R} = \int d\mathbf{v}'\, m\mathbf{v}' C_{ei}, \tag{35}$$

where p is the hydrostatic pressure, $\mathbf{q}$ the heat flux, $\mathbf{R}$ the friction force, and Q the thermal energy transfer. The viscosity for electrons is small and can be neglected in these equations. Equations (32)–(35) form a closed system of the fluid equations when $\mathbf{q}$ and $\mathbf{R}$ are given in terms of the plasma parameters, which requires the explicit solution of Eq. (31).

C. Zero order friction force

If the disturbance caused by the ions on the electrons is small, the displacement on the electron equilibrium distribution function is of the order of $\mathbf{U} = \mathbf{V} - \mathbf{V}_i$.[50] For such a shifted electron distribution function, $\mathbf{u}$ is independent of $\mathbf{V}_i$ and, therefore, the zeroth order friction force can be calculated from Eq. (28). From Eq. (35), knowing that the small perturbation corresponds to $U/\sqrt{T/m} \ll 1$, f_0 can be expanded in the power series of $\mathbf{U}$, and the friction force becomes

$$\mathbf{R}^{(0)} = \frac{2\pi e^2 e_i^2 n\lambda}{m}\int d\mathbf{v}\, m\mathbf{v}\frac{\partial}{\partial v_\nu}\left[V_{\mu\nu}\frac{\partial f_0^*(\mathbf{v}-\mathbf{U})}{\partial v_\mu}\right], \\ = -e^2 n^2 \eta_0 \mathbf{U}, \tag{36}$$

where τ is the relaxation time given in Eq. (30) and $\eta_0 = \sqrt{2/(5-3q)}A_q \pi^{\frac{3}{2}} m/(nq\tau e^2)$. The behaviour of $\mathbf{R}^{(0)}/\mathbf{R}^{(0)}_{Brag}$, where $\mathbf{R}^{(0)}_{Brag}$, correspondents to $q \to 1$ ($A_q = 1$ in this limit), as function of q, is depicted in Fig. 1. The initial decrease is explained by the reduction of the e-i collision frequency $\omega_{ei} \propto v^{-3}$ [see Eq. (29)] due to the increasing number of suprathermal electrons (faster electrons). After the minimum

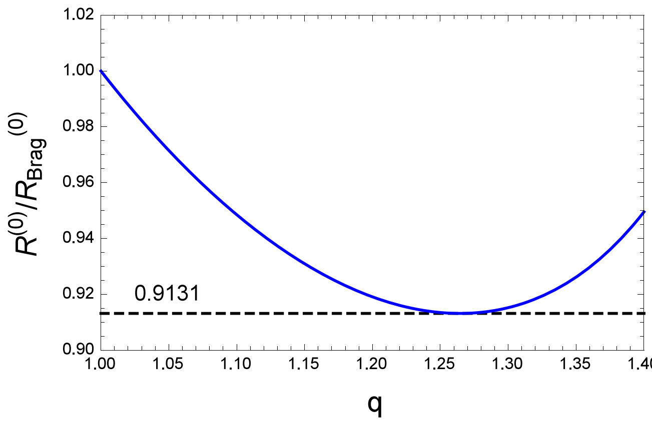

FIG. 1. The behavior of $\mathbf{R}^{(0)}/\mathbf{R}^{(0)}_{Brag}$ as a function of q. The decrease between $1 < q \leq 1.26$ accounts for the increasing number of suprathermal electrons, which reduces the cross section of the e-i collision. The growth for $q > 1.26$ is a consequence of the long-range correlations.

at $q \approx 1.26$, the subsequent increase in $\mathbf{R}^{(0)}$ is understood as a consequence of the long-range correlations, which are strong enough to overcome the reduction of ω_{ei} and increase the friction force, but without returning to the classical value.

V. CHAPMAN-ENSKOG METHOD

The solution of Eq. (31) by the Chapman-Enskog (CE) method is analogous to the classical procedure.[30,32,50] In the weak interaction limit, this solution is approximated by $f = f_0 + f_1$ and $f_1/f_0 \ll 1$. The direct substitution of f in the referred equation leads to

$$I_{ee}(f_0) + I'_{ei}(f_0) + \frac{e}{m}\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{37}$$

$$I_e(f_1) + \frac{e}{m}\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_1}{\partial \mathbf{v}} = \frac{df_0}{dt} + \mathbf{v} \cdot \nabla f_0 + \left(\frac{e\mathbf{E}'}{m} + \frac{d\mathbf{V}}{dt}\right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} + C'_{ei}(\mathbf{v} \cdot \mathbf{U} f_0), \tag{38}$$

where $I_e(f_1) = I_{ee}(f_1) + I_{ei}(f_1)$ are the linearised versions of the e-e and e-i collision operators and $C'_{ei}(\mathbf{v} \cdot \mathbf{U} f_0)$ is the small part of $C'_{ei}(f)$, all of them given in Appendix A. The separation of Eqs. (37) and (38) results from the ordination of the solutions imposed by the CE method.

The substitution of Eq. (15) into Eq. (37) proves that f_0 is the zero order solution, i.e., it represents the collisional equilibrium for the quasi-stationary states. The fluid equations calculated from Eq. (31) by taking the first three moments for $f = f_0$ are

$$\frac{dn}{dt} + n\nabla \cdot \mathbf{V} = 0, \tag{39a}$$

$$\frac{d\mathbf{V}}{dt} + \frac{e\mathbf{E}'}{m} = \frac{\mathbf{R}^{(0)} + \nabla p}{nm}, \tag{39b}$$

$$\frac{3}{2}\frac{dT}{dt} + T\nabla \cdot \mathbf{V} = 0. \tag{39c}$$

This set of equations, with exception of the explicit form of $\mathbf{R}^{(0)}$, Eq. (36), is equal to the zeroth order fluid equations found in the classical model.[32] Similar fluid equations have been found from other transport models within q-statistics.[25,51]

The first order solution of Eq. (38) is determined following the standard procedure of Refs. 30 and 50; namely, the elimination of the time derivatives in Eq. (38), with the help of Eqs. (39a)–(39c), and reorganization of the remaining terms in order to find approximative asymptotic solutions. In this process, the right-hand-side of Eq. (38) is written in terms of Jacobi polynomials,[52] whereas in the classical model associated Laguerre polynomials are used. This modification is necessary because the orthogonal properties of the former are better suited for the calculations of the relevant integrals involving power law distributions.

The straightforward calculation of the first order kinetic equation yields

$$\begin{aligned} I_e(f_1) &+ \frac{e}{m}\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_1}{\partial \mathbf{v}} \\ &= \left\{ \left(10\frac{1-q}{1+q} - \frac{5-3q}{1+q} L_1^{\frac{3}{2}}(x^2;q)\right) \cdot \nabla \ln T \right. \\ &\quad + \left[2\frac{1-q}{1+q}\left(10\frac{1-q}{5-3q} + L_1^{\frac{3}{2}}(x^2;q)\right)\right] \mathbf{v} \cdot \nabla \ln p \\ &\quad \left. + \frac{q\left(\mathbf{R}^{(0)} + \mathbf{R}^{(1)}\right) \cdot \mathbf{v}}{mT_q} \right\} \frac{f_0}{1-(1-q)x^2} + C'_{ei}(\mathbf{v} \cdot \mathbf{U} f_0), \end{aligned} \tag{40}$$

where $L_1^{3/2}(x^2;q) = P_1^{(3/2,1/(1-q))}(x^2) = -5/2 + (1+q)x^2/2$ is the first degree Jacobi polynomial and $\mathbf{R}^{(0)}$ is given in Eq. (36). In the above equation, the term proportional to $\nabla \ln p$ has no correspondent one in the classical model (i.e., $q \to 1$); it is an exclusive perturbation regarding the long tail distributions and originates from the non-cancellation between ∇f_0 and $\partial f_0/\partial \mathbf{v}$ due the power-law distributions. This new transport term has already been identified in the literature as the origin of a sort of anomalous collisional transport.[25,51]

The general solution of the linear equation, Eq. (40), can be written as a sum of the source terms on the right-hand-side, i.e.,

$$\begin{aligned} f_1 = \big[&A_T(x^2;q)\mathbf{v} \cdot \nabla \ln T + A_p(x^2;q)\mathbf{v} \cdot \nabla \ln p \\ &+ A_U(x^2;q)\mathbf{v} \cdot \mathbf{U}\big] f_0, \end{aligned} \tag{41}$$

where the A_j's are arbitrary functions.

The linear solution recovers the same bilinear relation between thermodynamic forces (perturbations) and associated (conductive) fluxes coupled by a transport coefficient, i.e., the well-known forms of the Fourier, Fick, and Ohm laws.[53] This is straightforwardly verified from Eq. (35) by the substitution of the general solution. In particular, the transport coefficient of the heat flux due $\mathbf{U}$ and the friction force due to ∇T are, respectively,

$$\begin{aligned} \alpha_U &= \frac{5-3q}{3n(1+q)} \int d\mathbf{v} v^2 L_1^{\frac{3}{2}}(v^2;q) A_U f_0, \\ \alpha_T &= -\frac{q}{3nT_q} \int d\mathbf{v} m v_\eta I_{ei}(v_\eta A_T), \end{aligned} \tag{42}$$

where I_{ei} is the linearised e-i collision operator given in Appendix A. This allows defining the thermal friction force, without loss of generality, by

$$\frac{q\mathbf{R}_w^{(1)}}{nT_q} = \alpha_w \mathbf{W}; \quad \mathbf{W} = (\nabla \ln T, \nabla \ln p), \tag{43}$$

where α_w is the corresponding transport coefficient to each perturbation $(w = T, p)$, and the first order friction force by

$$\mathbf{R}^{(1)} = -e^2 n^2 \eta_1 \mathbf{U}; \quad \eta_1 = \frac{1}{3e^2 n^2} \int d\mathbf{v} m v_\eta I_{ei}(v_\eta A_U). \tag{44}$$

Hence, the linear relations in Eqs. (41) and (43) account for the separation of Eq. (40) in a distinct equation for each perturbation in f_1 as follows:

$$I_e(A_T\mathbf{v}) - i\Omega\mathbf{v}A_T f_0 = \left[10\frac{1-q}{1+q} - \frac{5-3q}{1+q} L_1^{\frac{3}{2}}(x^2;q) + \alpha_T\right] \times \frac{\mathbf{v}f_0}{1-(1-q)x^2}, \tag{45}$$

$$I_e(A_p\mathbf{v}) - i\Omega\mathbf{v}A_p f_0 = \left[2\frac{1-q}{1+q}\left(10\frac{q-1}{5-3q} + L_1^{\frac{3}{2}}(x^2;q)\right) + \alpha_p\right] \times \frac{\mathbf{v}f_0}{1-(1-q)x^2}, \tag{46}$$

$$I_e(A_U\mathbf{v}) - i\Omega\mathbf{v}A_U f_0 = -\frac{qe^2n^2(\eta_0+\eta_1)}{nT_q} \times \frac{\mathbf{v}f_0}{1-(1-q)x^2} + C'_{ei}(\mathbf{v}f_0), \tag{47}$$

where the perpendicular and diamagnetic equations are coupled by A_w and α_w [the parallel direction $(\|)$ is obtained from the perpendicular taking $B \to 0$], $A_w = A_w^\perp + i\Omega A_w^\wedge$ and $\alpha_w = \alpha_w^\perp + i\Omega\alpha_w^\wedge$, $w = (T, p, U)$, $\Omega = eB/(mc)$ is the cyclotron frequency, η_0 is the friction coefficient from Eq. (36), and η_1 is the first order friction coefficient.

From the above set of equations and using the self-adjoint property of the collision operator from Eq. (A4) in Appendix A, the following relations between the transport coefficients can be proved:

$$\alpha_T = \alpha_U \equiv \alpha; \quad \begin{pmatrix}\kappa_p \\ \alpha_p\end{pmatrix} = 2\frac{q-1}{5-3q}\begin{pmatrix}\kappa_T \\ \alpha_T\end{pmatrix}, \tag{48}$$

where α_T and α_U are given by Eq. (42), and κ_T and κ_p are, respectively, the thermal conductivities due ∇T and ∇p calculated from Eq. (35) as

$$\begin{Bmatrix}\kappa_T \\ \kappa_p\end{Bmatrix} = \frac{2}{3}\frac{T_q}{1+q}\int d\mathbf{v} v^2 L_1^{\frac{3}{2}}(x^2;q) f_0 \begin{Bmatrix}A_T \\ A_p\end{Bmatrix}. \tag{49}$$

It is important to notice that the transport coefficients of the convective fluxes in all magnetic directions are included in the above expressions due the coupling of the kinetic equations; they follow the same representation of α in Eqs. (45)–(47).

The identity between the coefficients in Eq. (48) proves the Onsager reciprocity relations,[53] as verified in other formulations of the q-statistics.[54,55] In this context, the relations between κ_T and κ_p, and α_T and α_p represent extended reciprocity relations, where the transport coefficient of the convective flux due to ∇p is identified with those due ∇T.

The extended reciprocity relations enable the coupling of the gradient driven forces leading to the transport matrix

$$\begin{pmatrix}\dfrac{\mathbf{q}_j}{nT} \\ \dfrac{q\mathbf{R}_j}{nT_q}\end{pmatrix} = -\begin{pmatrix}\chi_{jT} & \alpha_j \\ \alpha_j & \dfrac{qne^2}{T_q}\eta\end{pmatrix}\begin{pmatrix}\nabla_j \ln\left(Tp^{2\frac{q-1}{5-3q}}\right) \\ -\mathbf{U}_j\end{pmatrix}, \tag{50}$$

where χ_{jT} is the heat diffusivity defined from $\kappa_{jT} = n\chi_{jT}$, $\eta = \eta_0 + \eta_1$, $\mathbf{q}_j$ and $\mathbf{R}_j$ are the total heat flux and friction force, respectively, and the index j stands for the parallel, perpendicular, and diamagnetic directions. In the above matrix, there is no diagonal term related to ∇p; therefore, this driving force behaves as a non-diagonal term and, eventually, transport particles and energy along or against ∇T, as the thermoelectric fluxes for instance. Hence, the A_p function could be defined up to a "±" sign; independently of this sign, the ordinary entropy production $\sigma_S \sim \mathbf{J}_a \cdot \mathcal{F}$, where $\mathbf{J}_a$ is the convective flux and $\mathcal{F}$ is the perturbation,[53] is always positive, whether the direction of the flow is towards or against ∇p. This result ensures that our model is consistent with the general framework of irreversible transport and the H-theorem, what is not true in previous models, where the possibility of negative transport coefficients allowed entropy sinks.[25,51]

VI. TRANSPORT COEFFICIENTS

One of the most distinctive solution methods of the classical transport model for magnetized plasmas was introduced by Braginskii;[50] basically, the first order solution of the kinetic equation is approximated by an asymptotic series of associated Laguerre polynomials and Maxwellian distributions. The main advantage of his method is using the orthogonality relations of such polynomials in order to, simultaneously, ensure the conditions of the CE-method and avoid the numerical solution of the equation.

Following the method of Braginskii, the A functions in Eqs. (45) and (47) are approximated by the asymptotic series of Jacobi polynomials

$$A_T = -\tau\sum_{k=1}^{\infty}\frac{a_k L_k^{\frac{3}{2}}(x^2;q)}{\left[1-(1-q)x^2\right]^{2+k}}, \tag{51}$$

$$A_U = \frac{m}{T_q}\sum_{k=1}^{\infty}\frac{a_k L_k^{\frac{3}{2}}(x^2;q)}{\left[1-(1-q)x^2\right]^{2+k}}, \tag{52}$$

where the coefficient $a_k = a_k^\perp + i\Omega a_k^\wedge$ is different for each series; the extended reciprocity relations obviate the necessity to solve the equation for A_p.

The orthogonality properties of the power law asymptotic expansions are readily verified when the orthogonal relations of the Jacobi polynomials[52] are employed in the conditions imposed by the CE-method[50]

$$\int d\mathbf{v}\left(1, m\mathbf{v}, \frac{mv^2}{2}\right) f_1 = 0, \tag{53}$$

where f_1 is given in Eq. (41) and the A functions are, respectively, given by and Eqs. (51) and (52).

Taking advantage of the orthogonality of the Jacobi polynomials, Eqs. (45) and (47) can be multiplied by an appropriated factor and integrated over $\mathbf{v}$ in order to obtain an infinity system of algebraic equations

$$\sum_{k=1}^{\infty} (c_{\ell k}^{ee} + c_{\ell k}^{ei} - i\Delta c_{\ell k}^{B}) a_k = c_\ell, \tag{54}$$

where $\Delta = \Omega\tau$ and the matrix elements c are given in Appendix B; namely, $c_{\ell k}^{ee}$, $c_{\ell k}^{ei}$, $c_{\ell k}^{B}$, and c_ℓ corresponding, respectively, to the integration of I_{ee}, I_{ei}, the magnetic term, and the source terms. The system of equations is similar to that found in the Braginskii formalism; its solutions provide the coefficients a_k of the asymptotic expansion. The asymptotic behavior of the coefficients in the perpendicular and diamagnetic directions in a strongly magnetized plasma ($\Delta \gg 1$) is proportional, respectively, to Δ^2 and Δ (the parallel direction does not depend on Δ). We also note that these components are the real and imaginary parts of $a_k = a_k^{\perp} + i\Omega a_k^{\wedge}$, as determined by the separation of the A functions in Eqs. (51) and (52).

Although this method allows the determination of all transport coefficients, we restrict the discussion only to the most relevant ones, namely, the perpendicular and parallel friction force coefficients, thermal conductivities, and the parallel thermoelectric coefficient. In terms of the asymptotic series given by Eqs. (51) and (52), the convective fluxes are

$$\mathbf{R} = \mathbf{R}^{(0)} + \mathbf{R}_U^{(1)} \approx \mathbf{R}^{(0)} + \sum_{k=1}^{\infty} \frac{nm}{\tau} r_U(q,k) a_k \mathbf{U}_{||}, \tag{55}$$

$$\mathbf{q}_U = \sum_{k=1}^{\infty} nT\, t_U(q,k) a_k \mathbf{U}_{||}, \tag{56}$$

$$\mathbf{q}_T = -\sum_{k=1}^{\infty} \frac{nT\tau}{n} c_T(q,k)\left[a_k\left(\nabla_{||}T + T\frac{q-1}{3-q}\nabla_{||}\ln n\right) + a_k'\left(\nabla_{\perp}T + T\frac{q-1}{3-q}\nabla_{\perp}\ln n\right)\right], \tag{57}$$

where we have neglected the perpendicular component of $\mathbf{R}^{(1)}$ in the first expression, since it is proportional to Δ^{-2}, whereas the same component in $\mathbf{R}^{(0)}$ is independent of Δ; we also have used $p = nT$ [see Eq. (35)] in the last equation. Then, r_U, t_U, and c_T are defined by

$$r_U(q,k) = \frac{2\pi^{\frac{3}{2}} A_q}{q^2}\left(\frac{2}{5-3q}\right)^{\frac{1}{2}} \times \int_0^{\infty} dx \frac{x L_k^{\frac{3}{2}}(x^2;q)\left[1-(1-q)x^2\right]^{\frac{q}{1-q}}}{\left[1-(1-q)x^2\right]^{1+k}}, \tag{58}$$

$$t_U(q,k) = \frac{2\pi A_q}{1+q}\frac{5-3q}{3} \times \int_0^{\infty} dx \frac{x^4 L_1^{\frac{3}{2}}(x^2,q) L_k^{\frac{3}{2}}(x^2,q)\left[1-(1-q)x^2\right]^{\frac{q}{1-q}}}{\left[1-(1-q)x^2\right]^{2+k}}, \tag{59}$$

$$c_T(q,k) = \frac{\pi(5-3q)A_q}{3(1+q)} \times \int_0^{\infty} dx \frac{x^2 L_1^{\frac{3}{2}}(x^2;q) L_k^{\frac{3}{2}}(x^2;q)\left[1-(1-q)x^2\right]^{\frac{q}{1-q}}}{(3-q)^{-1}\left(1-(1-q)x^2\right)^{2+k}}. \tag{60}$$

In order to determine the a_k's from Eq. (54), the infinity equation system has to be truncated; the degree of the asymptotic approximation is set by the number of remaining equations. All c coefficients in these equations are calculated analytically, except $c_{\ell k}^{ee}$, which is numerically obtained using the Monte Carlo method with random and stratified sampling.[56] This exception is imposed by the power-law distributions not allowing factorization in the integration variables as in the Braginskii formalism, for exponential-like distributions.

In short, the numerical calculations are performed according to a predetermined list of nine values in the range $q \in [1, 1.4]$, with 0.05 pace; each of the c's in Eq. (54) is evaluated analytically, whereas $c_{\ell k}^{ee}$ is numerically calculated by the Monte Carlo method. Then, the resulting system of equations is solved by matrix inversion and the limit $\Delta \gg 1$ is imposed. The transport coefficients are then calculated from Eqs. (55)–(57), where the analytical expressions of r_U, t_U, and c_T are also evaluated according to the predetermined list of values of q.

In order to simplify the representation of the transport coefficients, the numerical results obtained from Eqs. (55)–(57) are fitted by simple functions in accordance with Eq. (50), resulting in the following expressions:

$$\eta_{||} = \left(0.51 - \gamma_1(q-1)^{0.8}\left[1 - \gamma_2(5-3q)^{1.5}\right]\right)\frac{m}{ne^2\tau}, \tag{61}$$

$$\alpha_{||} = 0.71 + \theta_1\frac{(q-1)^{0.5}}{1+q}\left(1 - \theta_2(3-2q)^2\right), \tag{62}$$

$$\kappa_{||} = \frac{nT\tau}{m}\frac{3-q}{5-3q} \times (3.16 - 10\xi_1(q-1)^{1.5} \times [10 - \xi_2(9-5q)^{0.8}]), \tag{63}$$

$$\kappa_{\perp} = \frac{nT\tau}{m\Delta^2}\frac{3-q}{5-3q} \times \left(4.66 + \xi_3\frac{(q-1)^{1.25}}{(3-2q)^{0.5}}[10 - \xi_4(7-5q)^{1.1}]\right), \tag{64}$$

where the fitted coefficients are given in Table I. For $q \to 1$, these expressions recover the transport coefficients of the Braginskii model,[50]

$$\eta_{||Brag} = 0.51\frac{m}{ne^2\tau}, \quad \alpha_{||Brag} = 0.71, \tag{65}$$

TABLE I. Fitting values of the transport coefficients in Eq. (61).

γ_1	0.57	θ_1	4.85	ξ_1	1.46	ξ_3	2.69
γ_2	0.13	θ_2	0.95	ξ_2	6.07	ξ_4	4.75

$$\kappa_{||Brag} = 3.16\frac{nT\tau}{m}, \quad \kappa_{\perp Brag} = 4.66\frac{nT\tau}{m\Delta^2}. \tag{66}$$

The parallel friction force coefficient $\eta_{||}$ is calculated from Eq. (55) and shown in Fig. 2, together with the other coefficients. In order to obtain a reliable approximation for this coefficient, it is necessary to go up to the fifth order of the asymptotic approximation. Indeed, higher approximations are expected due to the slow convergence of the power-law asymptotic series when compared to the exponential character of the classical model. It turns out that $\eta_{||}$, therefore, $\mathbf{R}_{||}$, is a monotonically decreasing function of q, essentially because the weakening of the relaxation effect due to e-e collisions by suprathermal electrons, which enhance the long-tail of the distribution. The expression in Eq. (61) gives a result with less than $\pm 5\%$ relative error up to $q = 1.35$ and $\pm 8\%$ for $q = 1.4$.

The parallel thermoelectric coefficient $\alpha_{||}$ is calculated from Eq. (56) and the result is presented in Fig. 2. Due to the slow convergence, we have to follow the approximation up to the sixth order to ensure a reasonable convergence. These calculations are very sensitive to the numerical integration error; in particular, this is enhanced as the order of the approximation is increased, in special for $q \geq 1.2$. Such unsatisfactory variation is due to the high order polynomial solution of Eq. (54) in the sixth order approximation, which enhances the propagation of the numerical integration error. As a consequence, the relative error associated with this expression is smaller than $\pm 5\%$ only up to $q \leq 1.15$; it can vary from $\pm 7\%$ for $q = 1.2$ to 25% for $q = 1.4$.

In spite of the mentioned calculation difficulties, the dependence of $\alpha_{||}$ on q is quantitatively correct, as also pointed out in Appendix C. Indeed, its increase with q shown in Fig. 2 is understood according to the basic transport

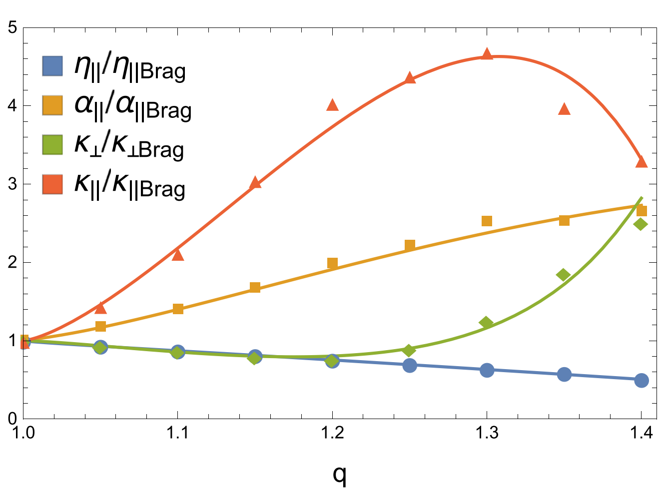

FIG. 2. Behaviour of the transport coefficient given in Eq. (61) as a function of q and normalized by the Braginskii values. The markers are numerical evaluations of the transport coefficients from Eqs. (55)–(57) and the lines the fitted polynomials in Eq. (61).

mechanism of the thermoelectric heat flux:[42,43] the net heat flow due to the difference between faster and slower electrons is enhanced by the increasing number of suprathermal electrons (faster electrons) flowing along **U**.

The parallel thermal conductivity $\kappa_{||}$, evaluated up to the sixth order of the asymptotic approximation, is given in Eq. (63). We warn that the precision of this expression deteriorates as $q \to 1.4$, as in the case of $\alpha_{||}$; it is smaller than $\pm 8\%$ for $q \leq 1.15$ and can vary from $\pm 15\%$ for $q = 1.2$ to $\pm 33\%$ for $q = 1.4$. Again, such an unsatisfactory variation for $q \geq 1.2$ is due to the high order polynomials required as the value of q increases, but the qualitative behaviour shown in Fig. 2 is correct (see Appendix C). In particular, the initial increase in $\kappa_{||}$ with q, due to the enhancement of the flux caused by the effect of suprathermal electrons, tends to saturate and eventually decrease as a consequence of the long-range correlations.

Interestingly, the calculation of the heat transport coefficient across the magnetic field is much less sensitive to the errors in the numerical calculation of $c^{ee}_{\ell k}$. In this case, the curves resulting from the different orders of the asymptotic approximation alternate with respect to an average one, so that the one corresponding to the sixth order is reasonably precise up to $q \approx 1.25$. The expression for $\kappa_{\perp}$ up to the sixth order of approximation is given by Eq. (64) and it is represented in Fig. 2. The relative error associated with this expression is smaller than $\pm 5\%$ for $q \leq 1.2$ and can vary from $\pm 6\%$ for $q = 1.25$ to $\pm 9\%$ for $q = 1.4$. It is evident from the figure that $\kappa_{\perp}$ initially decreases as the tail of the electron distribution function enlarges, up to $q \approx 1.2$. Above this value, $\kappa_{\perp}$ increases again, even beyond the value for the Braginskii model, corresponding to $q \to 1$.

Independent of the statistical distribution, the general behaviour of the ratio between $\kappa_{\perp}$ and $\kappa_{||}$ can be inferred from basic mechanical arguments, as $\kappa_{\perp}/\kappa_{||} \sim \omega^2/\Omega^2$, where ω is sort of a characteristic parallel collision frequency, i.e., the frequency of the scattering process presented by the collision operator without the dynamic effects of the evolution of f and the magnetic field. For the standard Braginskii model, it can be shown that $\omega \sim \tau^{-1}$, where τ is given by Eq. (30).[43] In the same sense, the normalized ratio $\kappa_{\perp}/\kappa_{||}$ obtained from Eqs. (63) and (64) is plotted in Fig. 3. The initial decrease can be attributed to the weakening of the scattering process due to the suprathermal electrons. Then, after reaching a minimum, the ratio starts to increase due to the effect of the long-range correlations. It is also important to note that the increase in $\kappa_{\perp}$ above the classical value is not a consequence of the scattering mechanism, but due to the new perturbation ∇p. In fact, despite $\kappa_{\perp}/\kappa_{||}$ staying always below the classical value, ∇p introduces the multiplicative term $(3-q)/(5-3q)$, which enhances both coefficients in Eqs. (63) and (64).

VII. HEAT FLUX IN THE SOLAR WIND

In the solar wind, the measurements of the field-aligned electron heat flux are not fully consistent with the predictions from the classical transport models, which is attributed to the suprathermal particles.[57] In spite of long-tail distribution

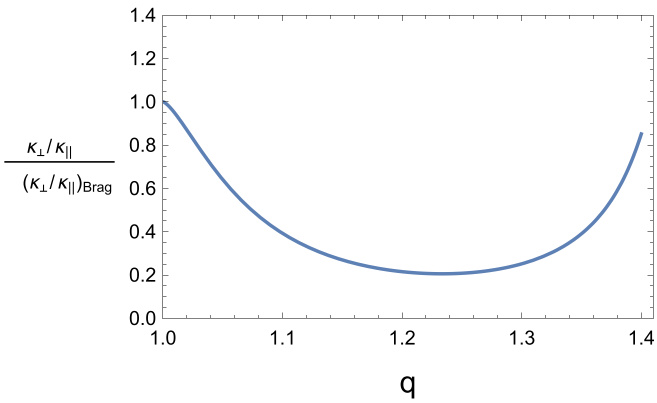

FIG. 3. $\kappa_\perp/\kappa_{||}$ as a function of q and normalized by the correspondent frequency of the classical model. The non-monotonicity is due the competition between the enhancement of the transport by suprathermal particles and the suppression of the local transport mechanism due to the long-range correlations.

functions reducing the effect of the collisions, the heat transport in solar winds is indeed dominated by them.[58] The deviations of the classical predictions from the data have successfully been modelled by the inclusion of a convective flux.[59] For instance, one of the most successful is the Hollweg model[60]

$$\mathbf{q}_{||} = -\kappa_{Brag}\nabla T + \frac{3}{2}nT\alpha_H\mathbf{V}, \tag{67}$$

where α_H is the Hollweg constant and $\mathbf{V}$ is the solar wind speed. In this model, the convective part of the total heat flux is understood as the consequence of the suprathermal electrons, which become appreciable when the wind velocity is of the order of the sound speed or when the plasma potential results from an electric field of the order of the Dreicer field.[61]

The total parallel heat flux is given by $\mathbf{q}_{||} = \mathbf{q}_{||U} + \mathbf{q}_{||T}$, calculated from Eqs. (56) and (57), respectively, and is written in terms of the $\kappa_\perp$ and α [Eqs. (63) and (62), respectively] as

$$\mathbf{q} = -\kappa\nabla T - 2\frac{q-1}{3-q}\kappa T\nabla\ln n + \alpha nT\mathbf{U}, \tag{68}$$

where the index $||$ was suppressed. Considering that Eq. (14) gives $\nabla\ln n = -e\mathbf{E}/T$, to the lowest order, and assuming that the electric field approaches the Dreicer field, $E \sim mv_T/(\tau e)$, which is consistent with $U = V \sim v_T$,[62,63] the above equation yields

$$\mathbf{q} = -\kappa\nabla T + \left[2\frac{q-1}{q-3}\frac{m\kappa}{nT\tau} + \alpha\right]nT\mathbf{V}, \tag{69}$$

where the square brackets defines the Hollweg constant

$$\alpha_H = \frac{2}{3}\left[2\frac{q-1}{q-3}\frac{m\kappa}{nT\tau} + \alpha\right]. \tag{70}$$

We note that using Eqs. (62) and (63) the expression of α_H depends only on constants and the value of q.

The observed κ-distributions correspond to $q = 1.1 - 1.5$ and $\alpha_H = 0.5 - 10$,[59,64,65] both consistent with the result of α_H, calculated from our model, and depicted in Fig. 4. In particular, typical expected values of the constant are $\alpha_H = 0.5 - 2$, for $q = 1.1 - 1.2$,[59–61] which are quite close to our predicted result indicated by the shaded area in the figure. Even more accurate results are expected from the direct numerical calculation using Eq. (68), instead of the approximated Eq. (69).

VIII. DISCUSSION AND CONCLUSION

In this work, we present a kinetic model for (quasi-) stationary plasmas far from thermodynamic equilibrium based on non-extensive statistics. Starting just from the definition of the S_q, we were able to derive the collisional equilibrium distribution function, the equilibrium temperature, and the kinetic equation with the consistent collisional operator for the weak interaction between charged particles. The derivation was kept as general as possible, ensuring all necessary conditions. This guarantees, despite our further restriction on the range $1 < q < 5/3$, that the model holds for the whole range of q $(-\infty < q < 5/3)$.

The existence of an generalized collision operator, where the long-tail distribution functions represent the "collisional equilibrium," could explain the persistence of the κ-distribution functions despite the collisions,[66] i.e., stationary states outside of the thermodynamic equilibrium. Indeed, if the dynamical equilibration, the origin of the long-tails, prevails over the relaxation toward the thermodynamic equilibrium, the collisional equilibrium would be approached by C_q, Eq. (24), rather by the ordinary Landau operator (Maxwellian distributions).

The fluid equations for the electrons in strongly magnetized plasmas were derived. These calculations were carried out by the Chapman-Enskog method, where the solutions are approximated as $f \approx f_0 + f_1$. In the zero order approximation, f_0, the friction force $\mathbf{R}^{(0)}$ is calculated in Eq. (36) and the result is depicted in Fig. 1. The non-monotonic behavior is understood as the competition between two effects: the decrease in the friction force as a consequence of $\omega_{\rm ei} \sim v^{-3}$ due to the increase number of suprathermal electrons; and, after the minimum at $q \approx 1.26$, the increasing due to the long-range correlations.

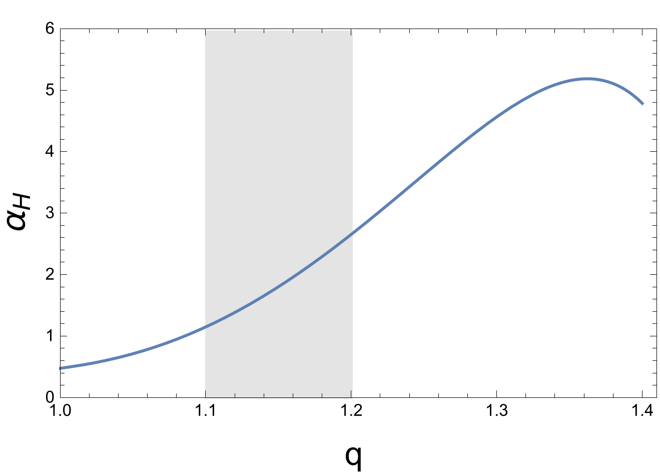

FIG. 4. Hollweg constant as a function of q including thermoelectric transport.

Using only general properties of f_1, the Onsager reciprocity relations were proved as well as extended reciprocity relations [see Eq. (48)] introduced. These new relations identify the transport coefficients associated with the fluxes of ∇p, a driven force exclusive to long-tail distributions, with those of ∇T. In particular, this formulation guarantees the positiveness of the ordinary entropy production, even if the flow due ∇p is along ∇T. This solves the problem of the negative transport coefficients in the previously q-kinetic models,[25,51] which correspond to a sink of entropy, in contradiction with the second law of the thermodynamics.

The explicit solution of f_1 [Eq. (41)] was approximated by the asymptotic series of Jacobian polynomials [Eqs. (51) and (52)]. Such choice takes advantage of the orthogonal properties of these polynomials to ensure the Chapman-Enskog conditions. This also enables the transformation of the first-order kinetic equation into a system of algebraic equations, which are used to determine the coefficients a_k of the asymptotic expansion and, consequently, the transport coefficients. Due to the power-law distributions, the asymptotic expansion has to be carried out until the fifth or the sixth order to guarantee reasonable accuracy for the transport coefficients. Except for the $\mathbf{R}_{||}$, all other calculated transport coefficients show a non-monotonic behavior (see Fig. 2). This is readily understood in Fig. 3, where $\kappa_\perp/\kappa_{||} \sim \omega^2/\Omega^2$ is depicted. The scattering process presented by the collision operator is enhanced by the suprathermal tail of the distribution function up to $q \approx 1.25$, where it starts to reduce due the long-range correlations, i.e., the dynamic effects which are responsible for the non-thermal stationary state.

As an example, the derived transport equations were applied to model heat transport in the solar wind. Using the formalism presented in this paper, we were able to present a justification for the empirical Hollweg model of heat transport in solar winds.[60] We rigorously identified that the convective part of the total heat flow originates from long-tail distributions, i.e., suprathermal electrons. The numerical values of the Hollweg constant (α_H) shown in Fig. 4 are consistent with the results found in the literature.[59–61]

In summary, in this work, for the first time, as far as we know, a self-consistent transport model in the non-extensive kinetic theory was presented. The general methodology was rigorously developed and the fluid equations in a strong magnetized plasma were obtained. The formalism was applied to help explaining empirical models used to describe parallel heat transport in space plasmas on the basis of the effect of suprathermal electrons. We hope that the theoretical findings presented here could help to improve the actually understanding and description of the effects of suprathermal electrons.

ACKNOWLEDGMENTS

This work was supported by the National Council of Scientific and Technological Developments (CNPq), under Grant Nos. 307984/2016-8 and 157898/2014-8. We would like to thank professors Dr. C. Tsallis, Dr. A. Elfimov, and Dr. G. P. Canal for useful discussions, and to the referee for pointing out a serious mistake in the original version of the article.

APPENDIX A: PROPERTIES OF THE COLLISION OPERATOR

In the weak interaction assumption, we have $f_1/f_0 \ll 1$, therefore, if $f_1 = f_0\psi$ then $\psi \ll 1$. Applying this condition for the e-e collision operator Eq. (24) in the weak interaction assumption, the linearised version of the collision operator is

$$\begin{aligned} I_{ee} &= C_{ee}(f_0, f_0'\psi') + C_{ee}(f_0\psi, f_0') \\ &= \frac{2\pi e^4\lambda}{m}\frac{5-3q}{2q^2}\frac{\partial}{\partial v_\nu} \times \int d\mathbf{v}' f_0 f_0' U_{\mu\nu} \\ &\quad \times \left[\frac{\partial}{\partial v_\mu}\left(\left[1-(1-q)x^2\right]\psi\right) - \frac{\partial}{\partial v_\mu'}\left[\left(1-(1-q)\psi'\right)\right]\right]. \end{aligned} \tag{A1}$$

The linearised version of the e-i collision operator is defined as the principal part of the collision operator [see Eqs. (27) and (28)]

$$I_{ei} = \frac{2\pi e^2 e_i^2 \lambda n}{m^2}\frac{\partial}{\partial v_\nu}\left[V_{\mu\nu}\frac{5-3q}{2q^2} f_0 \frac{\partial}{\partial v_\mu}\left[(1-(1-q)x^2\right]\psi\right], \tag{A2}$$

where $x^2 = mv^2/2T_q$ and $V_{\mu\nu} \equiv U_{\mu\nu}(\mathbf{v}_i = 0)$.

If we define

$$\hat{f} = \left[1-(1-q)x^2\right]\psi, \quad \hat{f}' = \left[1-(1-q)x\prime^2\right]\psi', \tag{A3}$$

the self-adjoint property of the collision operator[43] in Eq. (A1) can be proved when this equation is multiplied by $\hat{g}$ and then integrated over $\mathbf{v}$. Since in this circumstance both integration variables are dubbed, we can change $\mathbf{v} \to \mathbf{v}'$ and recover the same result. Hence, the self-adjoint property of the collision operator is expressed as

$$S_{ee}\left[\hat{f}, \hat{g}\right] = \frac{2\pi e^4\lambda}{m^2}\frac{5-3q}{2q^2}\int d\mathbf{v}d\mathbf{v}' f_0 f_0' U_{\mu\nu}\left(\frac{\partial \hat{g}}{\partial v_\nu} - \frac{\partial \hat{g}'}{\partial v_\nu'}\right), \\ \left(\frac{\partial \hat{f}}{\partial v_\mu} - \frac{\partial \hat{f}'}{\partial v_\mu'}\right) = S_{ee}\left[\hat{g}, \hat{h}\right], \tag{A4}$$

where we can see the symmetry between the exchange of functions $\hat{f}$ and $\hat{g}$.

The proof of such symmetric relation for Eq. (A2) is trivial, since the operator is linear in $\hat{f}$, and therefore

$$S_{ee}\left[\hat{f}, \hat{g}\right] = -\frac{2\pi e^4\lambda}{m^2}\frac{5-3q}{2q^2}\int d\mathbf{v}d\mathbf{v}' f_0 f_0' V_{\mu\nu}\frac{\partial \hat{g}}{\partial v_\nu}\frac{\partial \hat{f}}{\partial v_\mu}. \tag{A5}$$

In order to separate the small part of $C_{ei}'(f)$ in Eq. (38), we can add and subtract at the full e-i collision operator in Eq. (24) an ion distribution function shifted such that the mean ion velocity coincides with the mean electron velocity, just as in the Braginskii model[50]

$$C_{ei}(f, f_i) = C_{ei}'(f, f_i') + C_{ei}'(f, f_i - f_i'). \tag{A6}$$

The first term on the right-hand-side of the above equation is independent of $\mathbf{V}_i$; therefore, it is approximated by Eq. (28); the other term is the small term that can be approximated by the zeroth order solutions, i.e., f_0. Since this difference is small in this order, it can be expanded in power series of $\mathbf{U}$, which recovers the expression of e-i collision operator for $\mathbf{R}^{(0)}$ with the opposite sign, that is,

$$C'_{ei}(f, f_i - f'_i) = -\frac{2\pi e^2 e_i^2 n\lambda}{m^2}\int d\mathbf{v}\frac{\partial}{\partial v_\nu}\left[U_{\mu\nu}\frac{\partial f_0^*(\mathbf{v}-\mathbf{U})}{\partial v_\mu}\right],$$
$$= -C'_{ei}(\mathbf{v}\cdot\mathbf{U}f_0), \tag{A7}$$

which is also independent of $\mathbf{V}_i$. This expression is the same as in the Braginskii model, except by f_0^*.

APPENDIX B: COEFFICIENTS OF THE ALGEBRAIC EQUATION

The coefficients c of the integral transformation of the kinetic equations Eqs. (45) and (47) are equal, except by the term on the right side. Their expressions are

$$c_{\ell;U} = \int d\mathbf{v}\left(-\frac{\frac{4}{15}\frac{\tau}{2n}v_\xi L_\ell^{\frac{3}{2}}(x^2;q)}{\left[1-(1-q)x^2\right]^{\ell+1}}\right)C'_{ei}(v_\xi f_0); \tag{B1}$$

$$c_{\ell;T} = \frac{5-3q}{1+q}\frac{16\pi}{15}A_q\int_0^\infty dx\frac{x^4 L_1^{\frac{3}{2}}(x^2;q)L_k^{\frac{3}{2}}(x^2;q)}{\left[1-(1-q)x^2\right]^{-\frac{q}{1-q}+2+\ell}}, \tag{B2}$$

$$c_{\ell k}^m = \frac{8\pi}{15}A_q\int d\mathbf{x}\frac{x^2 L_\ell(x^2;q)L_k^{\frac{3}{2}}(x^2;q)}{\left[1-(1-q)x^2\right]^{-\frac{q}{1-q}+3+\ell+k}}, \tag{B3}$$

$$c_{\ell k}^{ei} = \int_0^\infty dx\frac{\frac{2A_q\pi^{\frac{3}{2}}}{5q^2}\left(\frac{2}{5-3q}\right)^{\frac{1}{2}}xL_\ell^{\frac{3}{2}}(x^2;q)L_k^{\frac{3}{2}}(x^2;q)}{\left[1-(1-q)x^2\right]^{-\frac{q}{1-q}+2+k+\ell}}, \tag{B4}$$

$$c_{\ell k}^{ee} = \int d\mathbf{v}\left(\frac{-\frac{4}{15}\frac{1}{n}\frac{m}{2T_q}\mathbf{v}L_k^{\frac{3}{2}}(x^2;q)}{\left[1-(1-q)x^2\right]^{1+\ell}}\right)I_{ee}(f_0, f_1), \tag{B5}$$

where the terms inside the parentheses in Eqs. (B1) and (B5) are the multiplicative factor used in the transformation of the kinetic equation. The collision operators used in $c_{\ell k}^{ee}$ and, in both, $c_{\ell k}^{ei}$ and $c_{\ell;U}$ are given, respectively, by Eqs. (A1) and (A2).

The integrals for all coefficients, but $c_{\ell k}^{ee}$, can be analytically evaluated. The integral of $c_{\ell k}^{ee}$ in numerically evaluated by the Monte Carlo method from Eq. (B5).

APPENDIX C: MONTE CARLO METHOD

The choice of the Monte Carlo method is due to the high dimension of the integral, which is not well approached by quadrature techniques.[56] The optimization routines of the method were also chosen in accordance with the computation performance. The random sampling with stratification of each axis in 4 subdivisions shown shorter time and smaller error in comparison with other routines. In particular, the adaptive techniques as well as the importance sampling were inefficient due to symmetries of the integrated function and the absence of regions of high accumulation (in the sampling phase of the method) when $q \to 1.4$.

The stratified Monte Carlo method divides the axis of the six integration variables in four parts, totalling 1296 subspaces, and samples approximately 8 millions of points in each of the 30 rounds of integration, for each of 9 values ranging over $q \in [1, 1.4]$ with pace of 0.05. The numerical error, for each of the 9 values, is then estimated by the standard deviation of the collection. As the error provided by the method, the standard deviation is understood as a probability range where the absolute numerical error could be found.[56] Since the convergence of the Monte Carlo method is $\sim 1/\sqrt{N}$, where N is the number of points sampled, if the N was quite large, the estimated value of the integral is well estimated by the mean, even if the relative error of the integral is inaccurate.

The sensibility of the transport coefficients with the numerical error of the integration is partly due to the high order polynomials resulting from the solution of the algebraic equations in Eq. (54). The other part is directly related to the form of the transport coefficient; for instance, $\alpha_{||}$, and $\kappa_{||}$, respectively, Eqs. (62) and (63), uses the same a_k, besides their error are different. Therefore, since the only error source is the numerical evaluation of $c_{\ell k}^{ee}$, this difference is caused by the propagation of the error in the definitions of these transport coefficients [see Eqs. (56) and (57)].

We also note that even a small imprecision could cause large differences due the mixing of $c_{\ell k}^{ee}$ with very different scales. For $q = 1.35$, the integration via Monte Carlo method results in $c_{16}^{ee} \approx 0.000039$ and $c_{66}^{ee} \approx 1.19101$. Therefore, an insignificant variation in c_{66}^{ee} could be enough to overcome the importance of the c_{16}^{ee}. In fact, this is was verified in the calculations of the sixth order approximations for the transport coefficients in Eq. (62), where the increase in the precision in the lower order matrix element, for example, c_{34}^{ee}, was more effective in reducing the overall error than if the precision is increased in the high order elements as c_{16}^{ee}.

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