Generalized Thermostatistical Description of Intermittency and Non-extensivity in Turbulence and Financial Markets

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Abstract. - We present a new framework for modeling the statistical behavior of both fully developed turbulence and short-term dynamics of financial markets based on the generalized non-extensive thermostatistics formalism. We also show that intermittency – strong bursts in the energy dissipation or clusters of high price volatility – and non-extensivity – anomalous scaling of usually extensive properties like entropy – are naturally linked by a single parameter q, from the non-extensive thermostatistics.

Scaling invariance plays a fundamental role in many natural phenomena and frequently emerges from some sort of underlying cascade process. A classical example is fully developed homogeneous isotropic three-dimensional turbulence, which is characterized by a cascade of kinetic energy from large forcing scales to smaller and smaller ones through a hierarchy of eddies. At the end of the cascade, the energy dissipates by viscosity, turning into heat. Recently, some authors [1-3] have studied the phenomenological relationship among financial market dynamics, scaling behavior and hydrodynamic turbulence. Particularly, Ghashghaie *et al.* [2] conjectured the existence of a temporal information cascade similar to the spatial energy cascade found in fully developed turbulence.

Traditionally, the properties of turbulent flows are studied from the statistics of velocity differences $v_r(x) = v(x) - v(x + r)$ at different scales r. As with other physical systems that depend on the dynamical evolution of a large number of nonlinearly coupled subsystems, the energy cascade in turbulence generates a spatial scaling behavior – power-law behavior with r – of the moments $\langle v_r^n \rangle$ of the probability distribution function (PDF) of v_r (the angle brackets $\langle\rangle$ denote the mean value of the enclosed quantity). For large values of the Reynolds number, which measures the ratio of nonlinear inertial forces to the linear dissipative forces within the fluid, there is a wide separation between the scale of energy input (integral scale L) and the viscous dissipation scale (Kolmogorov scale η). Though at large scales ($\sim L$) the PDFs are normally distributed, far from the integral scale they are strongly non-Gaussian and display wings fatter than expected for a normal process. This is the striking signature of the intermittency phenomenon. After publication of the Kolmogorov K62 refined similarity hypotheses [4], the problem of small scale intermittency became one of the central questions on isotropic turbulence. Over the past years several papers [5-12] have discussed intermittency and the so-called 'PDF problem'. Similar attempts [1,2,13] have been made to explain the same peculiar shape observed in the PDF of price changes $z_\tau = z(t) - z(t + \tau)$ at small time intervals.

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Based on the scaling properties of multifractals, a generalization of Boltzmann-Gibbs thermostatistics has been proposed [14-17] through the introduction of a family of non-extensive entropy functionals $S_q(p)$ with a single parameter q. These functionals reduce to the classical, extensive Boltzmann-Gibbs form as $q \to 1$. Optimizing $S_q[p]$ subject to appropriate constraints [16], we obtain the distribution

$$p_q(x) = [1 - \beta(1-q)x^2]^{1/(1-q)}/Z_q \quad . \tag{1}$$

The normalization factor, for $1 < q < 3$, is given by

$$Z_q \equiv \left[\frac{\pi}{\beta(q-1)}\right]^{1/2} \frac{\Gamma((3-q)/2(q-1))}{\Gamma(1/(q-1))} \quad .$$

In the limit of $q \to 1$, we recover the Gaussian distribution.

The above distribution, we claim, provides a simple and accurate model for handling the PDF problem. To show this, we stay in the context of fully developed turbulence ($x \equiv v_r$). From equation (1), we can easily obtain the second moment

$$\langle v_r^2 \rangle = \frac{1}{\beta(5-3q)} \quad , \tag{2}$$

and the flatness coefficient

$$K_r = \frac{\langle v_r^4 \rangle}{\langle v_r^2 \rangle^2} = \frac{3\,(5-3q)}{(7-5q)} \quad . \tag{3}$$

We remark that the flatness coefficient, which is directly related to the occurrence of intermittency, is solely determined by the parameter q.

At this point, if we assume [2,5,6] a scaling of the moments $\langle v_r^n \rangle$ of v_r as r^{ζ_n}, the variation with r of the PDF of the velocity differences and of its related moments can be completely determined. Particularly, we can obtain the functional forms of the flatness coefficient and the parameter q, respectively

$$K_r = K_L (r/L)^{\alpha} \tag{4}$$

and

$$q = \frac{15 - 7K_L(r/L)^{\alpha}}{9 - 5K_L(r/L)^{\alpha}} = \frac{15 - 7K_\eta(r/\eta)^{\alpha}}{9 - 5K_\eta(r/\eta)^{\alpha}} \quad , \tag{5}$$

where K_η is given by equation (4), $\alpha = \zeta_4 - 2\,\zeta_2$ and $K_L = 3$, the expected value for a Gaussian process. The correspondent expression for β can be derived similarly from equation (2). For a PDF normalized to its standard deviation (that is, with unit variance), we have $\beta = 1/(5-3q)$.

We note that in the limit of infinite Reynolds numbers, as $r \to 0$, K_r diverges (since $\alpha < 0$) while q tends to a finite limit, $q < 7/5$. This bound coincides with the one obtained by Boghossian [18] through a q-generalization of Navier-Stokes equations. Moreover, this limit implies that the second moment of distribution (1) will always remain finite, which is empirically expected from the phenomena here analyzed.

In order to account for the well known asymmetry of the velocity distributions, we may also consider $\beta = \beta_p$, for $v_r \geq 0$, and $\beta = \beta_m$, for $v_r < 0$. In this case, we have

$$\langle v_r^n \rangle = A_n(\beta_m, \beta_p)\, s^{n/2} \, \frac{\Gamma(\frac{n+1}{2})\Gamma(s - \frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(s - \frac{1}{2})} \quad , \tag{6}$$

where $s = 1/(q-1)$ and

$$A_n(\beta_m, \beta_p) = \frac{\beta_p^{-(n+1)/2} + (-1)^n\, \beta_m^{-(n+1)/2}}{\beta_p^{-1/2} + \beta_m^{-1/2}} \quad . \tag{7}$$

Accordingly, the value of parameters q, β_m and β_p, at a given scale r, are now determined from the second moment, the skewness ($S_r = \langle v_r^3 \rangle / \langle v_r^2 \rangle^{3/2}$) and the flatness coefficients.

We checked our model with turbulence statistics data taken from reference [2], provided by Chabaud *et al.* [9]. For this, we compared the experimental data with the theoretical predictions, for an skewness factor at the integral scale of $S_L = -0.4$, $L = 1$cm [9] and $\alpha = -0.10$ [5,19]. The results are displayed in Fig. 1a. A good agreement is observed through spatial scales spanning two orders of magnitude – from the neighborhood of the integral scale down to close to the Kolmogorov scale –, and for a range of up to 15 standard deviations, including the rare fluctuations in the tails of the distributions. Note that the solid lines in Fig. 1a have not been adjusted to the data through a free parameter, as in other models [9,6,7]. In the present case, the parameters q, β_m and β_p, that control the shape of the PDF in each scale, are uniquely determined from the scaling of $\langle v_r^2 \rangle$, S_r and K_r, obtained from equation (6).

The same approach adopted in turbulence can be straightforwardly applied (with $x \equiv z_\tau$ and z_τ scaling as τ^{ξ_n}) to model the statistics of price differences in financial markets, as far as the parameter α, S_L and the integral time scale τ_L – time span for which a convergence to a Gaussian process is found – are available. We tested our model with price changes data taken from reference [2], provided by Olsen & Associates. The results are displayed in Fig. 1b, for $\alpha = -0.16$, $S_L = -0.4$ and $\tau_L \simeq 2.3$ days. Again, we observe that the proposed model reproduces with good accuracy the statistics of price differences over all temporal scales.

Non-extensivity, a matter of speculation in some areas [20], is an essential feature of the generalized thermostatistics. If we suppose a scenario of a cascade of bifurcations with m levels, and scale the generalized expectation value of an observable O_q (the kinetic energy $\frac{1}{2}v_r^2$ of velocity differences, for example), averaged over a volume of size $V = \eta^3$ and normalized by Boltzmann constant, we have at the first level [21]

$$O_q(2V) = 2O_q(V) + 2(1-q_0)O_q(V)S_q(V) \tag{8}$$

and at level m

$$\begin{aligned} O_q(2^m V) &= 2O_q(2^{m-1}V) + 2(1-q_{m-1})O_q(2^{m-1}V)S_q(2^{m-1}V) \\ &\simeq 2^m O_q(V) + (1-q_{m-1})2^{m+C}O_q(V) \quad , \end{aligned} \tag{9}$$

where the higher order terms in $(1-q)$ have been neglected, with C being a constant to be determined later. Cascade processes are also described in terms of fractal or multifractals models [7,22-25]. Within these frameworks, in high Reynolds number turbulence, the energy dissipation is not uniformly distributed within the fluid but rather concentrated on subsets of non-integer fractal D_F dimension. This picture leads to a scaling behavior with dimensionality not equal to the dimension D of the embedding space. In this case, if we consider the cascade of bifurcations described above, we find

$$O_q(2^m V) = 2^{mD_F/D}O_q(V) \quad , \tag{10}$$

with $D = 3$. It follows immediately from equations (9) and (10) that

$$D_F \simeq \frac{D}{m} \left[\frac{\log(2^{-C} + 1 - q)}{\log(2)} + m + C \right] \quad . \tag{11}$$

At the top of the cascade, we have $q_{m-1} = 1$ and $D_F = D$. C is determined from the value of D_F at the bottom of the cascade. Writing $2^{m-1}\eta \equiv r$ and $q_{m-1} \equiv q(r)$, we get

$$D_F \simeq D \, \frac{\log((2^{-C} + 1 - q(r))\, 2^{C+1}\, r/\eta)}{\log(2\, r/\eta)} \quad , \tag{12}$$

where $q(r)$ is given by equation (5). For $C = -1$, at the bottom of the cascade ($r = \eta$), using the values of α and L previously specified, and $\eta = 0.022$mm [9], we get $D_F \simeq 2.37$. This value is in good agreement with the fractal dimension of interfaces in turbulent flows ($D_F = 2.35 \pm 0.05$), measured in different experimental contexts [26-29].

Equation (12), through the variation of parameter q, offers a quantitative picture of the transition from small-scale intermittent, non-extensive, fractal behavior to large-scale Gaussian, extensive homogeneity. This equation can also be exploited to estimate the structure functions exponents. Under the assumptions of the multifractal model [30], ζ_n is given by $\zeta_n = \inf_h [n\, h + D - D_F(h)]$, with $\zeta_3 = 1$, assuming that the turbulent flow possesses a range of scaling exponents $I = (h_{min}, h_{max})$. The existence of a corresponding range of dissipation scales η' extending from $\eta_{min} \sim L\, R^{-1/(1+h_{min})}$ to $\eta_{max} \sim L\, R^{-1/(1+h_{max})}$, where R is the Reynolds number at the integral scale, allows us to rewrite

$$\zeta_n = \inf_{\eta'} \; [n\, h(\eta') + D - D_F(\eta')] \quad , \tag{13}$$

where $D_F(\eta')$ is given by equation (12), with $r \equiv \eta'$. The resulting exponents ζ_n are in good agreement with experimental values [19,5], as shown in Table 1, for $n = 2, \ldots, 8$.

The above picture may also be applied to the information cascade, with $D = 1$. One main qualitative difference between the two processes is that, since there is nothing equivalent to viscous damping in the dynamics of speculative markets, the information cascade depth is only limited by the minimum time necessary to perform a trading transaction (roughly 1 min [1]). On the other hand, as in turbulence, the scaling exponents depend on the order of the moments in a nonlinear way. Figure 2 displays the comparison of the experimentally measured scaling exponents ξ_n [2] and the theoretical prediction. We found a good coincidence between our results and the experimental data, assuming an equivalent Reynolds number R of approximately 9000 and $\xi_2 = 1.06$ [3].

Summarizing, we described a simple and accurate framework for modeling the statistical behavior of both fully developed turbulence and short-term dynamics of financial markets based on the formalism of the generalized non-extensive thermostatistics. Within this framework, we have shown that intermittency and non-extensivity are naturally linked by parameter q, which represents an objective measure of small-scale intermittent, fractal behavior in turbulent cascades.

References

1. Mantegna, R.N. & Stanley, H.E. Scaling behaviour in the dynamics of an economic index. *Nature* **376**, 46-49 (1995).

2. Ghashghaie, S., Breymann, W., Peinke, J., Talkner, P. & Dodge, Y. Turbulent cascades in foreign exchange markets. *Nature* **381**, 767-770 (1996).

3. Mantegna, R.N. & Stanley, H.E. Turbulence and financial markets. *Nature* **383**, 587-588 (1996).

4. Kolmogorov, A. M. A refinement of previous hypotheses concerning the local structure of turbulence of a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **13**, 82-85 (1962).

5. Anselmet, F., Gagne, Y., Hopfinger, E. J. & Antonia, R. A. High-order velocity structure functions in turbulent shear flows. *J. Fluid Mech.* **140**, 63-89 (1984).

6. Castaing, B., Gagne, Y. & Hopfinger, E. J. Velocity probability density functions of high Reynolds number turbulence. *Physica D* **46**, 177-200 (1990).

7. Benzi, R., Biferale, L., Paladin, G., Vulpiani, A. & Vergassola, M. Multifractality in the statistics of the velocity gradients in turbulence. *Phys. Rev. Lett.* **67**, 2299-2302 (1991).

8. Sreenivasan, K. R. & Kailasnath, P. An update on the intermittency exponent in turbulence. *Phys. Fluids* **5**, 512-514 (1992).

9. Chabaud et al. Transition towards developed turbulence. *Phys. Rev. Lett.* **73**, 3227-3230, (1994).

10. Praskovsky, A. & Oncley, S. Measurements of the Kolmogorov constant and intermittency exponent at very high Reynolds number. *Phys. Fluids* **6**, 2886-2888 (1994)

11. Vassilicos, J.C. Turbulence and intermittency *Nature* **374**, 408-409 (1995).

12. Naert, A., Castaing, B., Chabaud, B., B. Hébral & Peinke, J. Conditional statistics of velocity fluctuations in turbulence. *Physica D* **113**, 73-78 (1998).

13. Taylor, S. J. *Math. Fin.* Modeling stochastic volatility: a review and comparative study. **4**, 183-204 (1994).

14. Tsallis, C. Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.* **52**, 479-487 (1988).

15. Tsallis, C., Sá Barreto, F.C. & Loh, E.D. Generalization of the Planck radiation law and application to the cosmic microwave background radiation. *Phys. Rev. E* **52**, 1447-1451 (1995).

16. Tsallis, C., Levy, S. V. F., Souza, A. M. C. & Maynard, R. Statistical-mechanical foundation of the ubiquity of Lévy distributions in nature. *Phys. Rev. Lett.* **75**, 3589-3593 (1995).

17. Lyra, M. L. & Tsallis, C. Non-extensivity and multifractality in low-dimensional dissipative systems *Phys. Rev. Lett.* **80**, 53-56 (1998).

18. Boghossian, B. M., Navier-Stokes equations for generalized thermostatistics. *Braz. J. Phys.* **29**, 91-107 (1999).

19. Benzi, R., Ciliberto, S., Baudet, C. & and Chavarria, C. R. On the scaling of three-dimensional homogeneous and isotropic turbulence. *Physica D* **80**, 385-398 (1995).

20. Maddox, J., When entropy does not seem extensive. *Nature* **365**, 103- (1993).

21. Boghossian, B. M., Thermodynamic description of the relaxation of two-dimensional Euler turbulence using Tsallis statistics. *Phys. Rev. E* **53**, 4754 (1996).

22. Mandelbrot, B. B. Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. *J. Fluid Mech.* **62**, 331-358 (1974).

23. Parisi, G. & Frisch, U. in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climatic Dynamics* (ed. Ghil, M., Benzi, R., & Parisi, G.) 84- (North-Holland, Amsterdam, 1985).

24. Paladin, G. & Vulpiani, A. Anomalous scaling laws in multifractal objects. *Phys. Rep.* **156**, 147-225 (1987).

25. Meneveau, C. & Sreenivasan, K. R. The multifractal nature of turbulent energy dissipation. *J. Fluid Mech.* **224**, 429-484 (1991).

26. Hentschel, H. G. E. & Procaccia, I. Relative diffusion in turbulent media: the fractal dimension of clouds. *Phys. Rev. A* **29**, 1461-1470, (1984).

27. Sreenivasan, K. R. & Meneveau, C. Fractal facets of turbulence. *J. Fluid Mech.* **173**, 357-386 (1986).

28. Sreenivasan, K. R., Ramshankar, R. & Meneveau, C. Mixing, entrainment and fractal dimensions of surfaces in turbulent flows. *Proc. R. Soc. Lond. A* **421**, 79-108 (1989).

29. Vassilicos, J. C. & Hunt, J. C. R. Fractal dimensions and spectra of interfaces with application to turbulence. *Proc. R. Soc. Lond. A* **435**, 505-534 (1991).

30. Frisch, U. *Turbulence* (Cambridge Univ. Press, Cambridge, UK, 1995).

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Captions

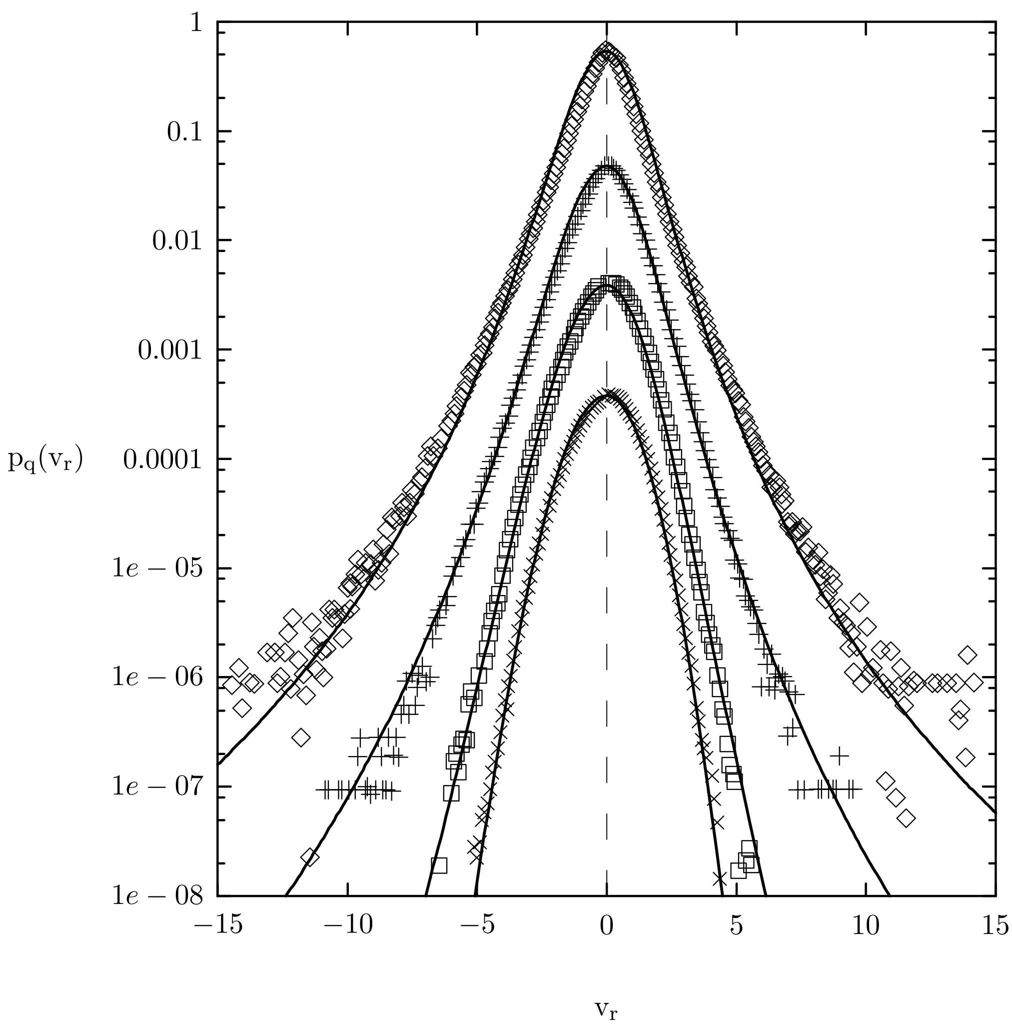
Table 1 Comparison of experimentally measured scaling exponents ζ_n and the theoretical prediction.

Figure 1(a) Data points: standardized probability distributions $p_q(v_r)$ of velocity differences $v_r(x) = v(x) - v(x+r)$ for spatial scales, from top to bottom, $r/\eta = 3.3$, 18.5, 138 and 325 (data taken from ref. [2], provided by Chabaud *et al.* [9]); Solid lines: proposed PDF model, equation (1) with $\beta = \beta_p$, for $v_r \geq 0$, and $\beta = \beta_m$, for $v_r < 0$ (for better visibility the curves have been vertically shifted with respect to each other).

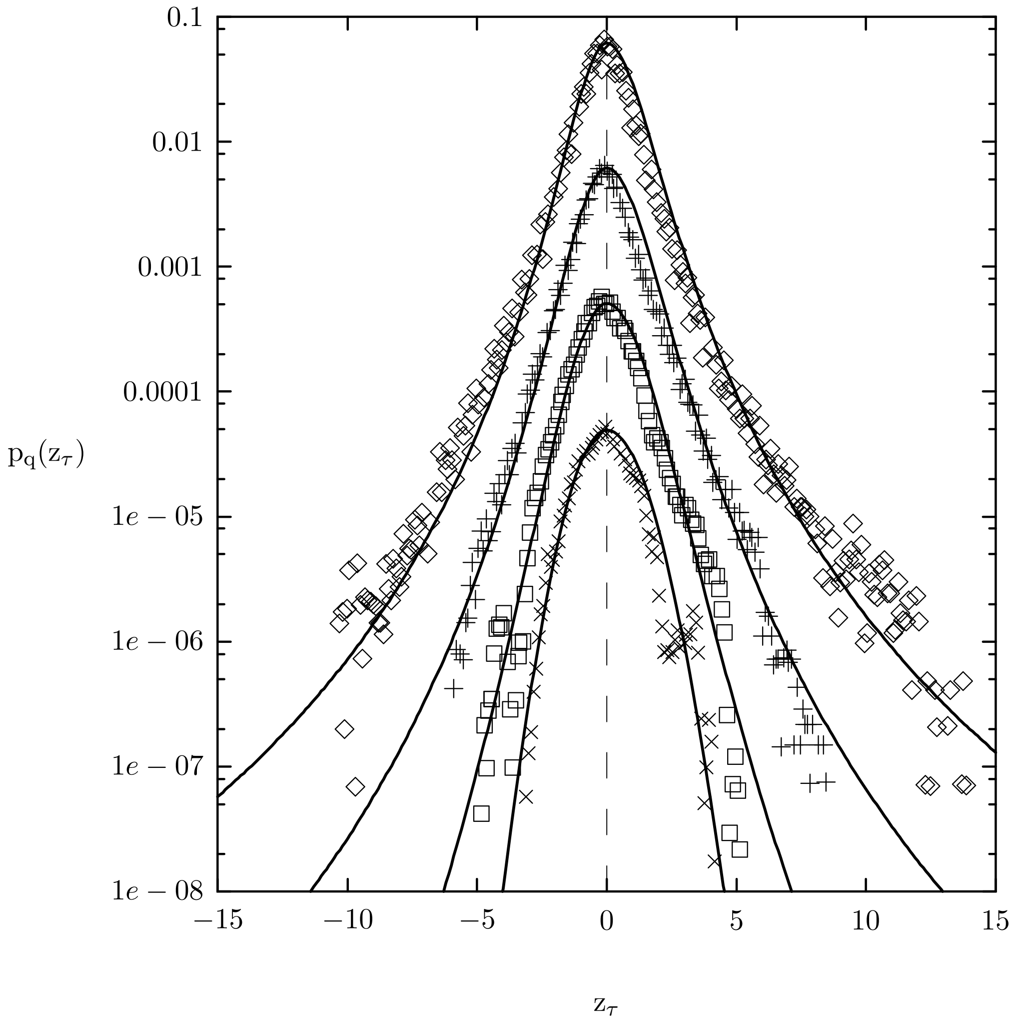
Figure 1(b) Data points: standardized probability distributions $p_q(z_\tau)$ of price differences $z_\tau(t) = z(t) - z(t+\tau)$ for temporal scales, from top to bottom, $\tau = 640\ s$, $5120\ s$, $40960\ s$ and $163840\ s$ (data taken from ref. [2], provided by Olsen & Associates); Solid lines: proposed PDF model, equation (1) with $\beta = \beta_p$, for $z_\tau \geq 0$, and $\beta = \beta_m$, for $z_\tau < 0$ (for better visibility the curves have been vertically shifted with respect to each other).

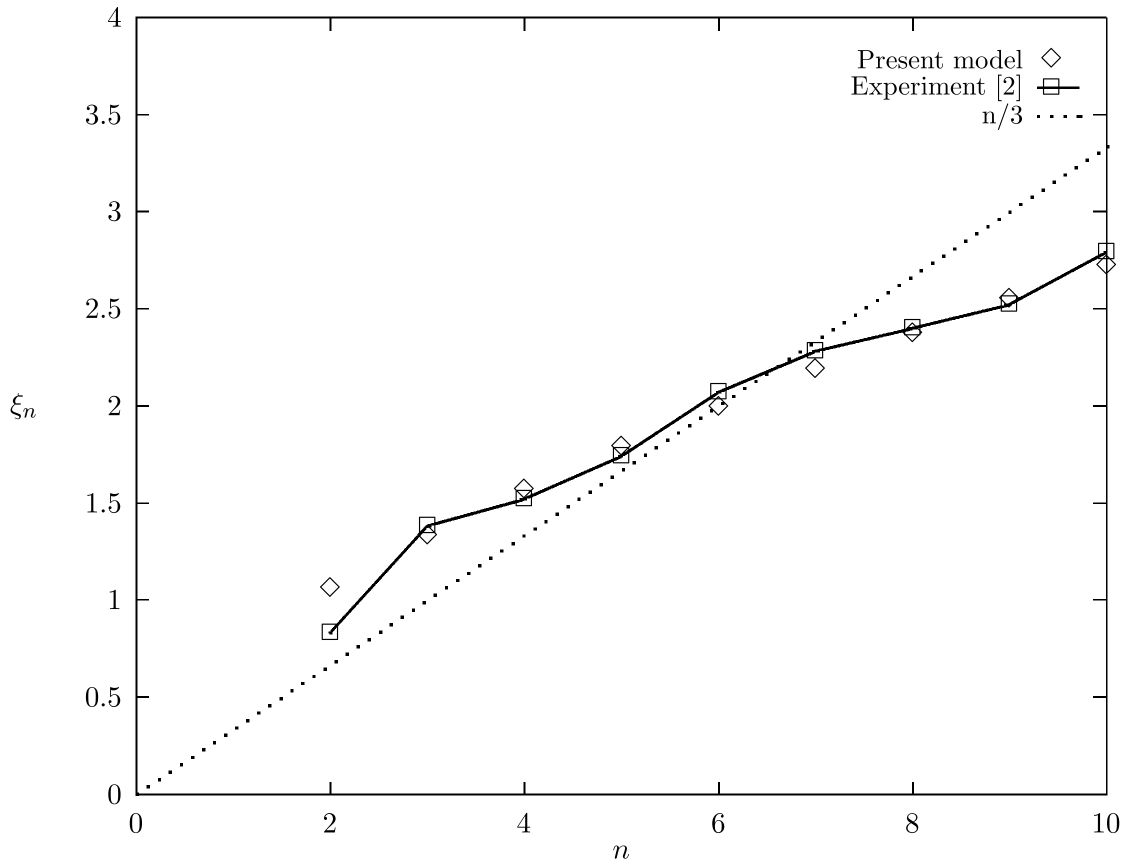
Figure 2 Comparison of theoretical and experimentally measured scaling exponents ξ_n.

Order n	Theory ζ_n	Experiment [19]	Experiment [5]
2	0.72	0.70	0.71
3	1.00	1.00	1.00
4	1.27	1.28	1.33
5	1.54	1.54	1.65
6	1.81	1.78	1.80
7	2.07	2.00	2.12
8	2.33	2.23	2.22



1
0.1
0.01
0.001
0.0001
1e − 05
1e − 06
1e − 07
1e − 08
−15
−10
−5
0
5
10
15
$p_q(v_r)$
v_r


0.1
0.01
0.001
0.0001
1e − 05
1e − 06
1e − 07
1e − 08
$p_q(z_\tau)$
−15
−10
−5
0
5
10
15
z_τ


Present model
Experiment [2]
n/3
ξ_n
n
0
0.5
1
1.5
2
2.5
3
3.5
4
0
2
4
6
8
10