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Extended gravitoelectromagnetism. III. Mercury's perihelion precession

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The motion of a test particle in planetary orbit around a central massive body is analyzed according to a novel, extended formulation of gravitoelectromagnetism. In particular, this extended form is used to calculate the correct anomalous precession of the perihelion of Mercury in Solar orbit as predicted by general relativity. The perihelion shift results from a balance between the gravitoelectric and gravitomagnetic forces.

I. INTRODUCTION

The perihelion precession of Mercury's orbit is one of the classical tests of relativity proposed by Einstein. The small observed deviation from the precession predicted by Newtonian theory, taking into account the interaction between the Sun and all the planets in the solar system, can be explained only introducing the relativistic corrections of general relativity [1–4].

In another aspect of the problem, the analogy between gravity and electromagnetism led to the development of gravitoelectromagnetism (GEM), motivated in great part by the calculations performed by Thirring and Lense of the gravitomagnetic (GM) effects produced by mass currents either in rotating spherical shells or rotating spherical bodies [5–7]. These calculations were performed according to the weak formulation of general relativity, but for given mass current profiles and neglecting terms of quadratic and higher orders in the test particle velocity components, that is, including only terms which are linear in the velocity v. As pointed out by Lense and Thirring [6], this first approximation to Einstein's theory excludes the perihelion perturbation which is obtained in the second approximation including terms quadratic in the velocity. The perturbations of the planetary motion considered by Lense and Thirring are due only to the rotation of the central body, without completely taking into account the relativistic motion of the test particle in the central field. If the central body is at rest, the relativistic correction to the perihelion precession is due to terms of order $\beta^2 = (v/c)^2$ in the planetary velocity v. In this case one must consider that, besides the relativistic corrections to the Newtonian motion, there is a gravitomagnetic contribution due to the mass current of the orbiting test particle, which is of the same order β^2 and must also be taken into account to give the correct result of general relativity, as will be shown in the present article. The Lense-Thirring effect corresponds to spin-orbit coupling while the correction to the precession of the perihelion must take into account the full orbit-orbit coupling effects. As also pointed out in the original work [6] it turns out that the Lense-Thirring effect in the solar system is much smaller than the full relativistic orbit-orbit correction.

Unfortunately, the traditional formulation of gravitoelectromagnetism as reviewed by Mashoon [8] and the gravitomagnetic effects described, for example, in the textbook by Moore [9] are based on the non relativistic Newtonian potential and on the first order gravitomagnetic Lorentz force, which depends linearly on the test particle velocity. As commented in the previous paragraph, this formulation fails to give the correct result for the full perihelion precession. The precession produced by the gravitomagnetic field due only to the rotation of a central body is very small and has the opposite sense with respect to the observed value. In fact, the Lense-Thirring effect due to the inner planet rotation was until recently too small to be measurable according to the review of Iorio et al. [10], but the final results of the Gravity Probe B experiment [11] confirmed the effect due to Earth's rotation at an accuracy of 19%. Despite this positive result of gravitomagnetic effects, it is clear that the correct calculation of the perihelion precession taking into account the full relativistic effects needs a revised formulation of gravitoelectromagnetism. This was recently accomplished by the extended gravitoelectromagnetic or hydrogravitoelectromagnetic theory reported in the first and second parts of a three parts work [12, 13]. The geodesic equation obtained through this theory is used in the present article to calculate the relativistic precession shift of Mercury.

The article is organized as follows. Section II introduces the geodesic equation of motion according to the extended gravitoelectromagnetic theory [13]. The gravitoelectromagnetic fields associated with the motion of an orbiting test particle are determined. Then, in Section III the equation for stable planetary motion is solved, showing how the balance between the gravitoelectric and gravitomagnetic fields defines the correct perihelion shift. The rate of change of the angular momentum is analyzed in Section IV, and Section V gives the final comments and conclusions.

II. GEODESIC EQUATION OF MOTION

Consider the planetary orbit of a mass m in the gravitational field ϕ_g produced by a central body of mass M. The central body M is assumed non rotating but the mass current density $\boldsymbol{j}_m$ due to the motion of the mass m produces a gravitomagnetic vector potential $\boldsymbol{A}_g$. The absence of intrinsic rotation (spin) excludes the Lense-Thirring effect. In the quasi-static regime (low frequency regime) there is no production of gravitational waves since the gravitoelectromagnetic force that drives the waves is negligible, that is, $\partial \boldsymbol{A}_g/\partial t \cong 0$ [13]. Actually, the gravitational waves are not excited because the retarded time effects are simply ignored. A simplified calculation of gravitational radiation by orbiting binaries, without general relativity, can be found in a recent publication [14].

According to the extended gravitoelectromagnetic formulation, the geodesic equation for a test mass in the vacuum region with negligible emission or absorption of gravitational waves is [13]

$$\frac{d\boldsymbol{v}}{dt} = -\left(1+\frac{v^2}{c^2}\right)\boldsymbol{\nabla}\phi_g + 4\frac{\boldsymbol{v}}{c}\cdot\left(\boldsymbol{\nabla}\phi_g\right)\frac{\boldsymbol{v}}{c} + 2\boldsymbol{v}\times\left(\boldsymbol{\nabla}\times\boldsymbol{A}_g\right) - 2\boldsymbol{v}\frac{\boldsymbol{v}}{c}\cdot\left(\boldsymbol{\nabla}\boldsymbol{A}_g\right)\cdot\frac{\boldsymbol{v}}{c}. \tag{1}$$

Now, in the quasi-static regime

$$\partial\phi_g/\partial t \cong 0 \quad \text{and} \quad \partial \boldsymbol{A}_g/\partial t \cong 0, \tag{2}$$

the scalar and vector potentials are given by the laws of Coulomb and Biot-Savart, respectively [12],

$$\begin{aligned}
\phi_g\left(\boldsymbol{r},t\right) &= -G\int\frac{\rho_m\left(\boldsymbol{r}',t\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r', \\
\boldsymbol{A}_g\left(\boldsymbol{r},t\right) &= -\frac{G}{c^2}\int\frac{\boldsymbol{j}_m\left(\boldsymbol{r}',t\right)}{\left|\boldsymbol{r}-\boldsymbol{r}'\right|}d^3r',
\end{aligned} \tag{3}$$

where G is the gravitational constant, ρ_m designates the mass density, and $\boldsymbol{j}_m = \rho_m\boldsymbol{v}_m$ the mass current density of a generic body of rest mass m. Since the generation of gravitational waves is neglected, ϕ_g and $\boldsymbol{A}_g$ are given in terms of the instantaneous mass density ρ_m and mass current density $\boldsymbol{j}_m$, disregarding time-lag effects. For a point mass, ρ_m and $\boldsymbol{j}_m$ can be taken as delta functions

$$\begin{aligned}
\rho_m\left(\boldsymbol{r},t\right) &= m\gamma_m\delta^3\left(\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)\right), \\
\boldsymbol{j}_m\left(\boldsymbol{r},t\right) &= m\gamma_m\boldsymbol{v}_m\delta^3\left(\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)\right),
\end{aligned} \tag{4}$$

where $\gamma_m = 1/\sqrt{1-v_m^2/c^2}$ is the Lorentz factor and $\boldsymbol{v}_m = d\boldsymbol{r}_m/dt$. Performing the volume integrations the potentials produced by the point mass are

$$\begin{aligned}
\phi_{g,m}\left(\boldsymbol{r},t\right) &= -Gm\frac{\gamma_m}{\left|\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)\right|}, \\
\boldsymbol{A}_{g,m}\left(\boldsymbol{r},t\right) &= -\frac{Gm}{c^2}\frac{\gamma_m\boldsymbol{v}_m}{\left|\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)\right|}.
\end{aligned} \tag{5}$$

Thus the induction gravitoelectromagnetic fields produced by a moving point mass m are

$$\begin{aligned}
\boldsymbol{E}_{g,m} &= -\boldsymbol{\nabla}\phi_{g,m} = -Gm\gamma_m\left(\frac{\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)}{\left|\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)\right|^3}\right), \\
\boldsymbol{B}_{g,m} &= \boldsymbol{\nabla}\times\boldsymbol{A}_{g,m} = \frac{Gm}{c^2}\gamma_m\boldsymbol{v}_m\times\boldsymbol{\nabla}\left(\frac{1}{\left|\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)\right|}\right) \\
&= -\frac{Gm}{c^2}\gamma_m\boldsymbol{v}_m\times\left(\frac{\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)}{\left|\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)\right|^3}\right) = \frac{\boldsymbol{v}_m\times\boldsymbol{E}_{g,m}}{c^2},
\end{aligned} \tag{6}$$

where the vector position $\boldsymbol{r}-\boldsymbol{r}_m\left(t\right)$ is directed from the point mass to the point where the fields are evaluated. For a point mass m passing near the body M the fields produced at the center of M are evaluated taking $\boldsymbol{r}=0$ ($\boldsymbol{r}_m$ becomes the outward pointing radial distance). Thus

$$\begin{aligned}
\boldsymbol{E}_{g,m} &= \frac{Gm\gamma_m}{\left|\boldsymbol{r}_m\right|^3}\boldsymbol{r}_m, \\
\boldsymbol{B}_{g,m} &= -\frac{\boldsymbol{E}_{g,m}\times\boldsymbol{v}_m}{c^2} = -\frac{Gm\gamma_m}{c^2\left|\boldsymbol{r}_m\right|^3}\boldsymbol{r}_m\times\boldsymbol{v}_m.
\end{aligned} \tag{7}$$

Note that this is the gravitoelectromagnetic field produced by m at the position of M without taking into account the gravitational pull of M. The central body M, which is assumed at rest, produces on its turn a gravitoelectric field at the position of m which is given by

$$\boldsymbol{E}_{g,M} = -\frac{GM}{|\boldsymbol{r}_m|^3}\boldsymbol{r}_m. \tag{8}$$

Since the point mass m in orbital motion experiences an outward acceleration $\boldsymbol{a} = -\boldsymbol{E}_{g,M}$ there is a Thomas precession [15]

$$\boldsymbol{\omega}_T = \frac{\gamma_m^2}{\gamma_m+1}\frac{\boldsymbol{a}\times\boldsymbol{v}_m}{c^2} = \frac{\gamma_m^2}{\gamma_m+1}\frac{GM}{c^2\,|\boldsymbol{r}_m|^3}\boldsymbol{r}_m\times\boldsymbol{v}_m. \tag{9}$$

This corresponds to the gravitomagnetic field produced by M at the position of m

$$\boldsymbol{B}_{g,M} = f_T\frac{GM}{c^2\,|\boldsymbol{r}_m|^3}\boldsymbol{r}_m\times\boldsymbol{v}_m. \tag{10}$$

Here the factor $f_T = \gamma_m^2/(\gamma_m+1) \sim 1/2$ is the kinematic Thomas factor ("Thomas half" for non relativistic velocities) arising from both a velocity change produced by the gravitational field and a rotation of the coordinates. This precession modifies the gravitomagnetic field produced by m on the center of M

$$\boldsymbol{B}_{g,m} = -f_T\frac{Gm\gamma_m}{c^2\,|\boldsymbol{r}_m|^3}\boldsymbol{r}_m\times\boldsymbol{v}_m = -f_T\frac{G}{c^2\,|\boldsymbol{r}_m|^3}\boldsymbol{L}_m \tag{11}$$
(GM field produced at central point $\boldsymbol{r}=0$ by the orbiting mass m),

where $\boldsymbol{L}_m = m\gamma_m\boldsymbol{r}_m\times\boldsymbol{v}_m$ is the angular momentum of m. Replacing $m\gamma_m$ by M and $\boldsymbol{r}_m$ by $-\boldsymbol{r}_m$ the same formula above gives the field produced by the central mass M at the frame of reference of the orbiting mass m (the orbiting velocity is the same)

$$\boldsymbol{B}_{g,M} = f_T\frac{GM}{c^2\,|\boldsymbol{r}_m|^3}\boldsymbol{r}_m\times\boldsymbol{v}_m = -f_T\frac{G}{c^2\,|\boldsymbol{r}_m|^3}\boldsymbol{L}_M \tag{12}$$
(GM field produced at orbit point $\boldsymbol{r}_m$ by the central mass M).

Consider the induction gravitoelectromagnetic fields produced by the central mass M on the mass m orbiting with a velocity $\boldsymbol{v}$ in the simplified notation ($\widehat{\boldsymbol{r}}$ is the unit vector pointing outwards from the center of M)

$$\begin{aligned} \boldsymbol{E}_g &= -\boldsymbol{\nabla}\phi_g = -GM\frac{\widehat{\boldsymbol{r}}}{r^2}, \\ \boldsymbol{B}_g &= \frac{f_T}{c^2}\boldsymbol{v}\times\boldsymbol{E}_g = -f_T\frac{GM}{c^2}\boldsymbol{v}\times\frac{\widehat{\boldsymbol{r}}}{r^2} = f_T\frac{G}{c^2r^3}\frac{M}{m\gamma}\boldsymbol{L}, \end{aligned} \tag{13}$$

where $\boldsymbol{L} = m\gamma\boldsymbol{r}\times\boldsymbol{v}$. The corresponding gravitoelectromagnetic potentials are

$$\begin{aligned} \phi_g &= -\frac{GM}{r}, \\ \boldsymbol{A}_g &= f_T\phi_g\frac{\boldsymbol{v}}{c^2} = -f_T\frac{GM}{c^2r}\boldsymbol{v}. \end{aligned} \tag{14}$$

Defining the four-momentum $mu_\mu = m\gamma(-c,\boldsymbol{v})$ and the four-vector potential $A^\mu = (\phi_g/c,\boldsymbol{A}_g)$, the interaction energy between the planetary mass m and the central mass M is given by

$$mu_\mu A^\mu = -m\gamma\phi_g + m\gamma\boldsymbol{v}\cdot\boldsymbol{A}_g = \frac{GMm\gamma}{r}\left(1-f_T\beta^2\right) = (2-\gamma)\frac{GMm\gamma}{r}. \tag{15}$$

The term $m\gamma\boldsymbol{v}\cdot\boldsymbol{A}_g$ corresponds to the direct gravitomagnetic interaction between m and M, given by the scalar product of the mass current of m with the vector potential produced by M (orbit-orbit interaction). The spin-orbit interaction due to intrinsic rotation of any of the two masses has been neglected. Accordingly, the Lense-Thirring effect is ignored. Moreover, possible spin-spin interactions have been also ignored. Note that the total gravitoelectromagnetic interaction energy changes sign for $\gamma > 2$ (actually this can not occur, as will be shown later). Note also that the gravitomagnetic force is given by

$$\boldsymbol{v}\times\boldsymbol{B}_g = -f_T\frac{GM}{r^2}\frac{v^2}{c^2}\widehat{\boldsymbol{v}}\times(\widehat{\boldsymbol{v}}\times\widehat{\boldsymbol{r}}) = f_T\frac{GM}{r^2}\frac{v^2}{c^2}\left[\widehat{\boldsymbol{r}}-(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\,\widehat{\boldsymbol{v}}\right], \tag{16}$$

which shows that the gravitomagnetic field opposes the gravitational attraction and has a braking effect on the planetary motion.

On substituting

$$\nabla \phi_g = \frac{GM\widehat{\boldsymbol{r}}}{r^2} \quad \text{and} \quad \boldsymbol{A}_g = -f_T \frac{GM\boldsymbol{v}}{c^2 r}, \tag{17}$$

the geodesic equation (1) for a test particle gives

$$\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= -\frac{GM}{r^2}\left[\left(1+\frac{v^2}{c^2}\right)\widehat{\boldsymbol{r}} - 4\frac{v^2}{c^2}\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\widehat{\boldsymbol{v}} \right. \\ &\left. +2f_T\frac{v^2}{c^2}\widehat{\boldsymbol{v}}\times\left(\widehat{\boldsymbol{v}}\times\widehat{\boldsymbol{r}}\right) + 2f_T\frac{v^4}{c^4}\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\widehat{\boldsymbol{v}}\right]. \end{aligned} \tag{18}$$

Using the vector relation

$$\widehat{\boldsymbol{v}}\times\left(\widehat{\boldsymbol{v}}\times\widehat{\boldsymbol{r}}\right) = \left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{r}}, \tag{19}$$

the geodesic equation becomes

$$\frac{d\boldsymbol{v}}{dt} + \frac{GM}{r^2}\widehat{\boldsymbol{r}} = \frac{GM}{r^2}\beta^2\left[\left(2f_T-1\right)\widehat{\boldsymbol{r}} + \left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\left(4-2f_T\left(1+\beta^2\right)\right)\widehat{\boldsymbol{v}}\right]. \tag{20}$$

This equation describes the three-dimensional motion of a test particle. Appendix VI defines a natural coordinate system for analyzing the general motion. However, this natural system is not needed in the present case, since only planar solutions will be considered in the next section.

III. PLANAR SOLUTION OF THE EQUATION OF MOTION

The scalar product of $(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}})$ with the above equation of planetary motion gives

$$\left(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}}\right)\cdot\frac{d\boldsymbol{v}}{dt} = 0. \tag{21}$$

This shows that stable planar motion in the plane formed by $\widehat{\boldsymbol{r}}$ and $\widehat{\boldsymbol{v}}$ is possible. Circular motion corresponds to $\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}} = 0$. This particular case is analyzed in Subsection III A, while the general elliptical orbit case is analyzed in Subsection III B.

A. Circular orbit

For $\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}} = 0$ the geodesic equation of motion (20) reduces to the equation for circular motion

$$\frac{d\boldsymbol{v}}{dt} + \frac{GM}{r^2}\widehat{\boldsymbol{r}} = \frac{GM}{r^2}\beta^2\left(2f_T-1\right)\widehat{\boldsymbol{r}}. \tag{22}$$

In general, the analysis of planar motion can be carried out in spherical coordinates (r,θ,φ), with the test particle velocity given by

$$\boldsymbol{v} = \frac{dr}{dt}\widehat{\boldsymbol{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}} + r\sin\theta\frac{d\varphi}{dt}\widehat{\boldsymbol{\varphi}} \tag{23}$$

and its squared amplitude by

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2. \tag{24}$$

The test particle acceleration becomes

$$\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\left(\frac{d\varphi}{dt}\right)^2\sin^2\theta\right]\widehat{\boldsymbol{r}} \\ &+ \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} - r\left(\frac{d\varphi}{dt}\right)^2\sin\theta\cos\theta\right]\widehat{\boldsymbol{\theta}} \\ &+ \left[r\frac{d^2\varphi}{dt^2}\sin\theta + 2r\frac{d\theta}{dt}\frac{d\varphi}{dt}\cos\theta + 2\frac{dr}{dt}\frac{d\varphi}{dt}\sin\theta\right]\widehat{\boldsymbol{\varphi}}. \end{aligned} \tag{25}$$

Hence the three components of the planetary orbit equation for circular motion (22) are

$$\begin{cases} \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 - r\left(\dfrac{d\varphi}{dt}\right)^2 \sin^2\theta &= -\dfrac{GM}{r^2}\left[1-\beta^2\left(2f_T-1\right)\right] \\ \dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) - r^2\left(\dfrac{d\varphi}{dt}\right)^2 \sin\theta\cos\theta &= 0 \\ \dfrac{d}{dt}\left(r^2\sin^2\theta\dfrac{d\varphi}{dt}\right) &= 0 \end{cases} \tag{26}$$

where

$$\begin{aligned} \beta^2 &= \frac{1}{c^2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2\right], \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}} \quad \text{and} \quad f_T = \frac{\gamma^2}{\gamma+1}. \end{aligned} \tag{27}$$

Assume a motion with the initial conditions $\theta = \pi/2$ and $d\theta/dt = 0$. At the initial time $t = 0$ the component equations reduce to

$$t = 0 \begin{cases} \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\varphi}{dt}\right)^2 &= -\dfrac{GM}{r^2}\left[1-\beta^2\left(2f_T-1\right)\right] \\ \dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) &= 0 \\ \dfrac{d}{dt}\left(r^2\dfrac{d\varphi}{dt}\right) &= 0 \end{cases} \tag{28}$$

Since this motion is stable, $d\theta/dt = 0$ and $\theta = \pi/2$ remain valid at all times (planar motion for $r \neq 0$). The spherical coordinates reduce to polar coordinates for $\theta = \pi/2$. The azimuthal component gives

$$\frac{d\varphi}{dt} = \frac{\ell}{r^2}, \tag{29}$$

where ℓ is a constant having the dimensions of a specific angular momentum (angular momentum per unit mass). This result gives a vanishing acceleration in the azimuthal direction (perpendicular to $\boldsymbol{r}$), so that the perihelion shift is eliminated or at least is undetectable. This is obviously true for a circular motion with azimuthal symmetry and conserved angular momentum. Now, consider the planetary orbit described by $r = r(\varphi)$. Then

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} \quad , \quad \frac{d^2r}{dt^2} = \frac{d^2r}{d\varphi^2}\left(\frac{d\varphi}{dt}\right)^2 + \frac{dr}{d\varphi}\frac{d^2\varphi}{dt^2}, \tag{30}$$

so that the radial and azimuthal components of the equation of motion yield

$$\begin{cases} \dfrac{dr}{d\varphi}\dfrac{d^2\varphi}{dt^2} + \left(\dfrac{d^2r}{d\varphi^2} - r\right)\left(\dfrac{d\varphi}{dt}\right)^2 + \dfrac{GM}{r^2} \\ \qquad = \dfrac{GM}{c^2r^2}\left(2f_T-1\right)\left[\left(\dfrac{dr}{d\varphi}\right)^2 + r^2\right]\left(\dfrac{d\varphi}{dt}\right)^2 \\ \dfrac{d^2\varphi}{dt^2} = -\dfrac{2}{r}\dfrac{dr}{d\varphi}\left(\dfrac{d\varphi}{dt}\right)^2 \end{cases} \tag{31}$$

These two equations can be combined in the form

$$\begin{aligned} &\left[\frac{d^2r}{d\varphi^2} - \frac{2}{r}\left(1 + \frac{GM}{2c^2r}\left(2f_T-1\right)\right)\left(\frac{dr}{d\varphi}\right)^2\right. \\ &\left. -\frac{GM}{c^2}\left(2f_T-1\right) - r\right]\left(\frac{d\varphi}{dt}\right)^2 + \frac{GM}{r^2} = 0. \end{aligned} \tag{32}$$

Substituting the solution $d\varphi/dt = \ell/r^2$ gives an autonomous nonlinear second-order differential equation for the orbit

$$\frac{d^2r}{d\varphi^2} - \frac{2}{r}\left(1 + \frac{GM}{2c^2r}\left(2f_T-1\right)\right)\left(\frac{dr}{d\varphi}\right)^2 - \frac{GM}{c^2}\left(2f_T-1\right) - r + \frac{GM}{\ell^2}r^2 = 0. \tag{33}$$

The time evolution along the orbit is determined by

$$t = \frac{1}{\ell} \int_{0}^{\varphi} r\left(\varphi'\right)^2 d\varphi'. \tag{34}$$

The radial and time coordinates can be normalized by a radial distance a and an angular frequency $\omega_0 = \ell/a^2$ as follows

$$\xi = \frac{r}{a} \quad , \quad \tau = \frac{\ell}{a^2} t = \omega_0 t. \tag{35}$$

Thus

$$\left\{ \begin{array}{rcll} \dfrac{d^2\xi}{d\varphi^2} - \dfrac{2}{\xi}\left(1 + \dfrac{\epsilon}{2\xi}\left(2f_T - 1\right)\right)\left(\dfrac{d\xi}{d\varphi}\right)^2 & & & \\ -\epsilon\left(2f_T - 1\right) - \xi + \alpha\xi^2 & = & 0 & \text{orbit equation} \\ \dfrac{d\varphi}{d\tau} & = & \dfrac{1}{\xi^2} & \text{time evolution} \end{array} \right. \tag{36}$$

where the relativistic correction factor ϵ and the coefficient α are dimensionless parameters defined by

$$\epsilon = \frac{GM}{c^2 a} \quad , \quad \alpha = \frac{GMa}{\ell^2} = \frac{GM}{\omega_0^2 a^3}. \tag{37}$$

The above equations should describe a circular orbit with $\xi = 1$, so that ϵ and α must satisfy the following consistency condition

$$\alpha = 1 + \epsilon\left(2f_T - 1\right). \tag{38}$$

Note that for $\xi = 1$ $(r = a)$

$$\beta^2 = \frac{1}{c^2}\left[\left(\frac{dr}{d\varphi}\right)^2 + r^2\right]\left(\frac{d\varphi}{dt}\right)^2 = \frac{\omega_0^2 a^2}{c^2}\left[\left(\frac{d\xi}{d\varphi}\right)^2 + \xi^2\right]\left(\frac{d\varphi}{d\tau}\right)^2 = \frac{\omega_0^2 a^2}{c^2} \tag{39}$$

corresponds to the constant velocity of the test particle, and

$$T_0 = \frac{2\pi a^2}{\ell} = \frac{2\pi}{\omega_0} = \frac{2\pi a}{c\beta} \tag{40}$$

gives the period of the complete orbit. Hence, the consistency condition can be written as

$$\frac{\epsilon}{\beta^2} = 1 + \epsilon\left(2f_T - 1\right) \tag{41}$$

or in the equivalent form

$$\frac{\gamma^2 - 1}{2\left(2 - \gamma\right)\gamma^2 - 1} = \epsilon. \tag{42}$$

This condition leads to a cubic equation relating γ to ϵ. The real value of γ is given by

$$\gamma = \frac{1}{6\epsilon}\left[2\left|1 - 4\epsilon\right| \cos\left(\frac{1}{3}\arccos\left(\frac{-1 + 2\epsilon\left(6 + \epsilon\left(3 + 5\epsilon\right)\right)}{\left|1 - 4\epsilon\right|^3}\right)\right) - 1 + 4\epsilon\right]. \tag{43}$$

Figure 1 shows the range of possible values of γ as a function of ϵ. The real value can be represented as a power series for $\epsilon \ll 1$

$$\gamma \cong 1 + \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} - \frac{\epsilon^3}{16} - \frac{77\epsilon^4}{128} - \frac{163\epsilon^5}{256} + \frac{399\epsilon^6}{1024} + \frac{3991\epsilon^7}{2048} + \ldots \tag{44}$$

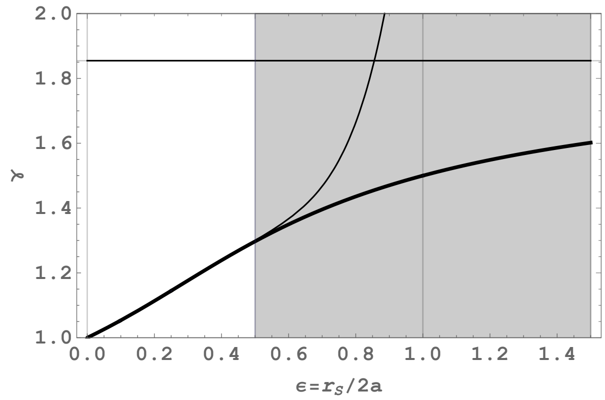

FIG. 1. The thick line shows the value of γ as a function of ϵ for circular motion. The thin curved line shows the series expansion for $\epsilon \ll 1$ and the thin horizontal line indicates the asymptotic limit for $\epsilon \to \infty$. The dark shaded area to the right of $\varepsilon = 1/2$ is excluded since it corresponds to orbital radii a smaller than the Schwarzschild radius $r_S = 2GM/c^2$ of the central mass.

Note that in the weakly relativistic limit the Thomas factor is $f_T \cong 1/2 + 3\epsilon/8$. Conversely, the asymptotic limit for $\epsilon \gg 1$ is given by

$$\lim_{\epsilon\to\infty} \gamma = \frac{2}{3}\left[1 + 2\cos\left(\frac{1}{3}\arccos\left(\frac{5}{32}\right)\right)\right] \cong 1.85464. \tag{45}$$

This limit indicates that the critical value $\gamma = 2$ for which the total gravitoelectromagnetic interaction energy changes sign cannot occur, as expected. Now, to first order in ϵ one has $\gamma \cong 1 + \epsilon/2$, and the orbiting velocity of the test particle becomes

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \cong \sqrt{\epsilon} \tag{46}$$

so that

$$v_0 \cong c\sqrt{\epsilon} = \sqrt{\frac{GM}{a}} \quad \text{and} \quad T_0 = \frac{2\pi a}{c\beta} \cong 2\pi\sqrt{\frac{a^3}{GM}} \tag{47}$$

give, respectively, the orbit velocity and the period of the Kepler circular orbit in the non relativistic limit.

From the almost circular orbit of Venus listed in Table I one obtains an orbital period $T_0 = 224.7 \times 24 \times 3600 = 1.9414 \times 10^7\,\mathrm{s}$. Using the Sun's mass $M = 1.9891 \times 10^{30}\,\mathrm{kg}$ and the gravitational constant $G = 6.67408 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ the non relativistic calculated orbital period is $T_0 = 2\pi\sqrt{a^3/(GM)} = 1.9411 \times 10^7\,\mathrm{s}$. The calculated nearly circular orbit velocity is $v_0 = \sqrt{GM/a} = 3.5026 \times 10^4\,\mathrm{m\,s^{-1}}$. With the speed of light $c = 2.99792458 \times 10^8\,\mathrm{m\,s^{-1}}$, the relativistic correction factor is $\epsilon = GM/\left(c^2 a\right) = 1.36503 \times 10^{-8}$.

For a neutron star with a mass $2M$ and a 11000 m radius the relativistic correction factor at its surface is $\epsilon \cong 0.269$, so that the consistency condition is well described by the power series expansions in the allowed region to the left of the Schwarzschild limit $\varepsilon \le 1/2$. Of course, the neutron star rotation would introduce spin-orbit interaction in the test particle motion (Lense-Thirring effect).

B. Elliptical orbit

The general equation of motion (20) is here repeated:

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} + \frac{GM}{r^2}\widehat{\boldsymbol{r}} &= \frac{GM}{r^2}\beta^2\left[(2f_T - 1)\,\widehat{\boldsymbol{r}}\right.\\ &\quad \left.+ (\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\left(4 - 2f_T\left(1+\beta^2\right)\right)\widehat{\boldsymbol{v}}\right].\end{aligned} \tag{48}$$

Venus data	
Semimajor orbit axis a	1.08209475×10^{11} m
Orbit eccentricity e	0.00677672
Mean orbit velocity v_0	3.50206×10^{4} m s^{-1}
Volume V	$9.28415345893\times10^{20}$ m^3
Mass m	4.867320×10^{24} kg
Length of year	224.7 Earth days

TABLE I. Values taken from NASA's Solar System Exploration/Planets/Venus website.

The radial and azimuthal components of the planetary orbit equation are, respectively,

$$\left\{\begin{array}{rcl} \frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 + \frac{GM}{r^2} & & \\ & = & \frac{GM}{c^2r^2}\left[\left(3-2f_T\beta^2\right)\left(\frac{dr}{dt}\right)^2 - \left(1-2f_T\right)r^2\left(\frac{d\varphi}{dt}\right)^2\right] \\ \frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) & = & \frac{4GM}{c^2}\left(1-\frac{f_T}{2}\left(1+\beta^2\right)\right)\frac{dr}{dt}\frac{d\varphi}{dt} \end{array}\right. \tag{49}$$

For a planetary orbit described by $r = r(\varphi)$ the two components of the equation of motion become

$$\left\{\begin{array}{rcl} \frac{dr}{d\varphi}\frac{d^2\varphi}{dt^2} + \left(\frac{d^2r}{d\varphi^2} - r\right)\left(\frac{d\varphi}{dt}\right)^2 + \frac{GM}{r^2} & & \\ & = & \frac{GM}{c^2r^2}\left[\left(3-2f_T\beta^2\right)\left(\frac{dr}{d\varphi}\right)^2 - \left(1-2f_T\right)r^2\right]\left(\frac{d\varphi}{dt}\right)^2 \\ & & \text{radial component} \\ \frac{d^2\varphi}{dt^2} & = & -\frac{2}{r}\frac{dr}{d\varphi}\left[1-\frac{GM}{c^2r}\left[2-f_T\left(1+\beta^2\right)\right]\right]\left(\frac{d\varphi}{dt}\right)^2 \\ & & \text{azimuthal component} \end{array}\right. \tag{50}$$

These equations can be combined in the form

$$\begin{array}{l} \left[\frac{d^2r}{d\varphi^2} - \frac{2}{r}\left(1-\left(1-2f_T\right)\frac{GM}{2c^2r}\right)\left(\frac{dr}{d\varphi}\right)^2\right. \\ \left. -r\left(1-\left(1-2f_T\right)\frac{GM}{c^2r}\right)\right]\left(\frac{d\varphi}{dt}\right)^2 + \frac{GM}{r^2} = 0. \end{array} \tag{51}$$

In general, the azimuthal equation of motion can be written as

$$\frac{d^2\varphi}{dt^2} = F\left(\varphi, \frac{d\varphi}{dt}\right)\left(\frac{d\varphi}{dt}\right)^2. \tag{52}$$

Dividing by $d\varphi/dt$ and integrating, the solution can be written in implicit form

$$\begin{array}{l} \ln\frac{d\varphi}{dt} = C_1 \int\limits_{\varphi(0)}^{\varphi(t)} F\left(\varphi_1, \frac{d\varphi_1}{dt}\right)d\varphi_1 \\ \Longrightarrow \frac{d\varphi}{dt} = \left(\frac{d\varphi}{dt}\right)_0 \exp\left[\int\limits_{\varphi(0)}^{\varphi(t)} F\left(\varphi_1, \frac{d\varphi_1}{dt}\right)d\varphi_1\right]. \end{array} \tag{53}$$

Hence

$$\begin{aligned} \frac{d\varphi}{dt} &= \left(\frac{d\varphi}{dt}\right)_0 \exp\left(-2\int\limits_{r[\varphi(0)]}^{r[\varphi(t)]}\frac{dr_1}{r_1}\right)\exp\left(\frac{4GM}{c^2}\int\limits_{r[\varphi(0)]}^{r[\varphi(t)]}\frac{dr_1}{r_1^2}\right) \\ &\quad\times\exp\left(-\frac{2GM}{c^2}\int\limits_{r[\varphi(0)]}^{r[\varphi(t)]} f_T(r_1)\left[1+\beta^2(r_1)\right]\frac{dr_1}{r_1^2}\right) \\ &= \frac{\ell}{r^2}\exp\left(\frac{4GM}{c^2 a}\frac{r-a}{r}\right) \\ &\quad\times\exp\left(-\frac{2GM}{c^2}\int\limits_{r[\varphi(0)]}^{r[\varphi(t)]} f_T(r_1)\left[1+\beta^2(r_1)\right]\frac{dr_1}{r_1^2}\right), \end{aligned} \tag{54}$$

where $a = r[\varphi(0)]$ is the semimajor axis (characteristic radius of the orbit) and $\ell = a^2 (d\varphi/dt)_0$ corresponds to the conserved specific angular momentum in the non relativistic case. The last term in the right-hand side of the azimuthal velocity involves an integral of factors which depend on $f_T = \gamma^2/(\gamma+1)$ and on

$$\beta^2 = \frac{1}{c^2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] = \frac{1}{c^2}\left[\left(\frac{dr}{d\varphi}\right)^2 + r^2\right]\left(\frac{d\varphi}{dt}\right)^2. \tag{55}$$

The integral can be simplified taking into account that in planetary motion the squared relativistic velocity $\beta^2 = v^2/c^2$ is very small so that

$$f_T\left(1+\beta^2\right) = \left(2\gamma^2-1\right)/(\gamma+1) \sim 1/2 + 7\beta^2/8 \tag{56}$$

is essentially constant. Assuming constant $f_T \sim 1/2$ and neglecting the higher order terms in β^2 (the β^2 factor multiplying f_T corresponds to the $O(\beta^4)$ term in the geodesic equation) the azimuthal component of the equation of motion gives

$$\frac{d\varphi}{dt} \cong \frac{\ell}{r^2}\exp\left[\frac{4GM}{c^2 a}\left(1-\frac{f_T}{2}\right)\left(\frac{r-a}{r}\right)\right]. \tag{57}$$

Note that the dominant factor in the azimuthal velocity corresponds to the conservation of angular momentum multiplied by a small relativistic oscillation around the semimajor orbital axis. Since there is no azimuthal symmetry the instantaneous angular momentum is not conserved in the relativistic case. Only the average value in a complete orbit is conserved. This approximate solution can be replaced in the radial equation, resulting in an autonomous nonlinear second-order differential equation for the orbit

$$\begin{aligned} \frac{d^2 r}{d\varphi^2} - \frac{2}{r}\left(1-(1-2f_T)\frac{GM}{2c^2 r}\right)\left(\frac{dr}{d\varphi}\right)^2 - r\left(1-(1-2f_T)\frac{GM}{c^2 r}\right) \\ +\frac{GM}{\ell^2} r^2 \exp\left[-\frac{8GM}{c^2 a}\left(1-\frac{f_T}{2}\right)\left(\frac{r-a}{r}\right)\right] \cong 0. \end{aligned} \tag{58}$$

The time evolution along the orbit is determined by

$$t \cong \frac{1}{\ell}\int\limits_0^{\varphi} r(\varphi')^2 \exp\left[-\frac{4GM}{c^2 a}\left(1-\frac{f_T}{2}\right)\left(\frac{r(\varphi')-a}{r(\varphi')}\right)\right] d\varphi'. \tag{59}$$

As shown in Subsection 3.1, the radial and time coordinates can be normalized by the radial distance a and an angular frequency $\omega_0 = (d\varphi/dt)_0 = \ell/a^2$

$$\xi = \frac{r}{a} \quad , \quad \tau = \frac{\ell}{a^2} t = \omega_0 t. \tag{60}$$

The orbit equation in normalized form becomes

$$\begin{aligned} \frac{d^2\xi}{d\varphi^2} - \frac{2}{\xi}\left(1-(1-2f_T)\frac{\epsilon}{2\xi}\right)\left(\frac{d\xi}{d\varphi}\right)^2 - \xi\left(1-(1-2f_T)\frac{\epsilon}{\xi}\right) \\ +\alpha\xi^2\exp\left[-2(4-2f_T)\left(\frac{\xi-1}{\xi}\right)\epsilon\right] \cong 0, \end{aligned} \tag{61}$$

and the time evolution along the orbit is given by

$$\tau \cong \int_0^{\varphi} \xi\left(\varphi'\right)^2 \exp\left[-\left(4-2f_T\right)\left(\frac{\xi\left(\varphi'\right)-1}{\xi\left(\varphi'\right)}\right)\epsilon\right] d\varphi', \tag{62}$$

where $\epsilon = GM/\left(c^2 a\right)$ is the previously defined relativistic correction factor, and the dimensionless parameter $\alpha = GM/\left(\omega_0^2 a^3\right)$ will be related below with both ϵ and the orbit eccentricity.

The differential orbit equation can be simplified introducing a change in the normalized radial coordinate

$$\xi = \frac{1}{\eta} \quad , \quad \frac{d\xi}{d\varphi} = -\frac{1}{\eta^2}\frac{d\eta}{d\varphi} \quad , \quad \frac{d^2\xi}{d\varphi^2} = \frac{2}{\eta^3}\left(\frac{d\eta}{d\varphi}\right)^2 - \frac{1}{\eta^2}\frac{d^2\eta}{d\varphi^2}, \tag{63}$$

so that

$$\frac{d^2\eta}{d\varphi^2} + \left(2f_T - 1\right)\epsilon\left(\frac{d\eta}{d\varphi}\right)^2 \cong f\left(\eta\right), \tag{64}$$

where

$$f\left(\eta\right) = \alpha \exp\left[2\left(4-2f_T\right)\left(\eta-1\right)\epsilon\right] - \eta\left[1 + \left(2f_T-1\right)\epsilon\eta\right]. \tag{65}$$

In the relativistic case the planetary orbit is governed by a Rayleigh equation for nonlinear oscillations. The loss of kinetic energy during the revolution is compensated by a source of energy which is determined by the system itself. With the substitution $w\left(\eta\right) = \left(d\eta/d\varphi\right)^2$ the orbit equation reduces to the first-order nonlinear equation

$$\frac{dw}{d\eta} + 2\left(2f_T-1\right)\epsilon w\left(\eta\right) \cong 2f\left(\eta\right), \tag{66}$$

whose general solution is (C =constant)

$$w\left(\eta\right) \cong 2e^{-2(2f_T-1)\epsilon\eta}\left(C + \int_{\eta(\varphi_0)}^{\eta} e^{2(2f_T-1)\epsilon\eta_1} f\left(\eta_1\right) d\eta_1\right). \tag{67}$$

Therefore, the solution of the original orbit equation can be written in implicit form

$$\varphi - \varphi_0 \cong \pm \int_{\eta(\varphi_0)}^{\eta(\varphi)} \frac{d\eta_2}{\sqrt{2e^{-2(2f_T-1)\epsilon\eta_2}\left(C + \int_{\eta(\varphi_0)}^{\eta_2} e^{2(2f_T-1)\epsilon\eta_1} f\left(\eta_1\right) d\eta_1\right)}}. \tag{68}$$

Imposing the initial conditions at the aphelion

$$\eta\left(\varphi_0\right) = \eta_a, \quad \left.\frac{d\eta}{d\varphi}\right|_{\varphi_0} = 0, \tag{69}$$

the integration constant vanishes ($C = 0$) so that

$$\frac{d\eta}{d\varphi} \cong \pm\sqrt{2e^{-2(2f_T-1)\epsilon\eta}\int_{\eta_a}^{\eta} e^{2(2f_T-1)\epsilon\eta_1} f\left(\eta_1\right) d\eta_1}. \tag{70}$$

The position of the perihelion η_p with respect to the aphelion η_a is defined by a second solution of the condition $d\eta/d\varphi = 0$. This condition can be used to define the value of the constant coefficient α in terms of the positions η_a and η_p of the aphelion and the perihelion, respectively,

$$\alpha \cong \frac{\displaystyle\int_{\eta_a}^{\eta_p} e^{2(2f_T-1)\epsilon\eta_1}\left[1 + \left(2f_T-1\right)\epsilon\eta_1\right]\eta_1 d\eta_1}{\displaystyle e^{-2(4-2f_T)\epsilon}\int_{\eta_a}^{\eta_p} e^{6\epsilon\eta_1} d\eta_1}. \tag{71}$$

Thus

$$\begin{aligned}\alpha &\cong \frac{3\epsilon\left(\eta_p^2 - e^{2(1-2f_T)\epsilon(\eta_p-\eta_a)}\eta_a^2\right)e^{-2(4-2f_T)\epsilon(\eta_p-1)}}{1-e^{-6\epsilon(\eta_p-\eta_a)}} \\ &\cong \frac{\eta_p+\eta_a}{2} - \frac{\epsilon}{2}\left[(5-4f_T)\left(\eta_p^2+\eta_a^2\right) - 2\left(4-2f_T\right)\left(\eta_p - \eta_p\eta_a + \eta_a\right)\right] + O\left(\epsilon^2\right).\end{aligned} \tag{72}$$

The positions η_a and η_p are defined in terms of the orbit eccentricity e (recall that $\eta = a/r$):

$$\eta_a = \frac{1}{1+e} \quad , \quad \eta_p = \frac{1}{1-e}. \tag{73}$$

Hence

$$\begin{aligned}\alpha &\cong \frac{3\epsilon}{(1-e)^2}\frac{\exp\left(-\frac{2(4-2f_T)\epsilon}{1-e}\right)}{1-\exp\left(-\frac{12e\epsilon}{1-e^2}\right)}\left[1-\left(\frac{1-e}{1+e}\right)^2\exp\left(-\frac{4\left(2f_T-1\right)e\epsilon}{1-e^2}\right)\right] \\ &\cong \frac{1}{1-e^2} - \frac{1+9e^2-2f_T\left(1+3e^2\right)}{\left(1-e^2\right)^2}\epsilon + O\left(\epsilon^2\right).\end{aligned} \tag{74}$$

Therefore, the specific angular momentum parameter ℓ is given by

$$\begin{aligned}\ell &= \sqrt{\frac{GMa}{\alpha}} \\ &= \sqrt{GMa\left(1-e^2\right)}\left(1+\frac{1+9e^2-2f_T\left(1+3e^2\right)}{2\left(1-e^2\right)}\epsilon + O\left(\epsilon^2\right)\right).\end{aligned} \tag{75}$$

In summary, the coefficient α is determined in terms of both the relativistic factor ϵ and the positions η_a and η_p of the aphelion and the perihelion, respectively, by

$$\alpha \cong \frac{3\epsilon\left(\eta_p^2 - e^{2(1-2f_T)\epsilon(\eta_p-\eta_a)}\eta_a^2\right)e^{-2(4-2f_T)\epsilon(\eta_p-1)}}{1-e^{-6\epsilon(\eta_p-\eta_a)}}, \tag{76}$$

or equivalently in terms of the orbit eccentricity e, and the relativistic factor ϵ, as listed previously. The orbit is governed by the previously derived autonomous differential equation (64)

$$\begin{aligned}&\frac{d^2\eta}{d\varphi^2} + \left(2f_T-1\right)\epsilon\left(\frac{d\eta}{d\varphi}\right)^2 \\ &\cong \alpha\exp\left[2\left(4-2f_T\right)(\eta-1)\epsilon\right] - \eta\left[1+\left(2f_T-1\right)\epsilon\eta\right].\end{aligned} \tag{77}$$

Figure 2 shows a numerical solution of the above equation. It illustrates the perihelion shift for a fictitious planet with an elliptical orbit of eccentricity $e = 1/2$ and large orbital velocity corresponding to a relativistic correction factor $\epsilon = 1/500$. The oblique line indicates the final position of the aphelion predicted by general relativity to first order in ϵ after 50 complete revolutions of the planet starting at the position indicated by the extreme point on the right-hand side.

An approximate solution of the relativistic orbit differential equation can be obtained using a perturbation method. A change of the independent angular variable

$$\varphi = \psi\left(1+\epsilon\psi_1+\ldots\right) \tag{78}$$

in the orbit equation yields

$$\begin{aligned}\frac{d^2\eta}{d\psi^2} + \left(2f_T-1\right)\epsilon\left(\frac{d\eta}{d\psi}\right)^2 \cong \left(1+\epsilon\psi_1+\ldots\right)^2\{&\alpha\exp\left[2\left(4-2f_T\right)(\eta-1)\epsilon\right] \\ &-\eta\left[1+\left(2f_T-1\right)\epsilon\eta\right]\}.\end{aligned} \tag{79}$$

Replacing η by

$$\eta\left(\psi\right) = \eta_0\left(\psi\right) + \epsilon\eta_1\left(\psi\right) \tag{80}$$

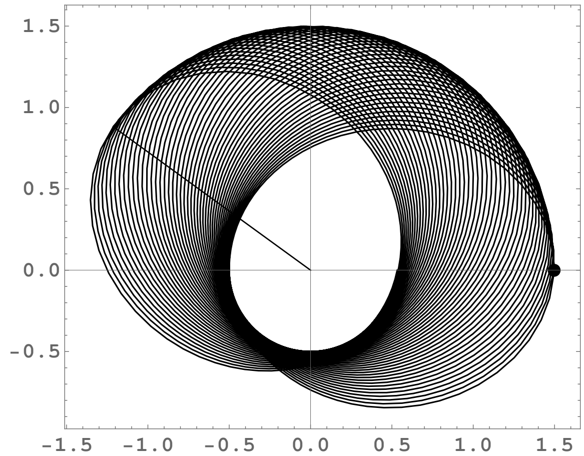

FIG. 2. Illustration of the perihelion shift for a fictitious planet in elliptical orbit. The orbit is characterized by the semimajor axis $a = 1$, eccentricity $e = 1/2$ and large relativistic correction factor $\epsilon = 1/500$. The orbit starts at the aphelion $a\,(1+e)$, lying at the extreme point on the right-hand side of the horizontal axis. A total of 50 revolutions is displayed resulting in a 145.751° shift. The oblique line indicates the approximate final position predicted by general relativity, which to first order in ϵ corresponds to a 144° shift.

an expansion to first order in ϵ gives a system of equations for the leading terms of the series:

$$\left\{\begin{array}{rcl} \dfrac{d^2\eta_0}{d\psi^2}+\eta_0 &=& \alpha \\ \dfrac{d^2\eta_1}{d\psi^2}+\eta_1 &=& 2\,(\alpha-\eta_0)\,\psi_1-(2f_T-1)\left(\dfrac{d\eta_0}{d\psi}\right)^2 \\ && +\left[2\,(4-2f_T)\,\alpha-(2f_T-1)\,\eta_0\right]\eta_0-2\,(4-2f_T)\,\alpha \end{array}\right. \tag{81}$$

The zero order solution with initial conditions $\eta_0\,(\psi_0)=\eta_a$ and $d\eta_0/d\psi|=0$ gives the Kepler orbit

$$\eta_0\,(\psi)=\alpha-(\alpha-\eta_a)\cos\,(\psi-\psi_0)\,. \tag{82}$$

Substituting this solution in the first order differential equation gives

$$\begin{array}{r} \dfrac{d^2\eta_1}{d\psi^2}+\eta_1 \cong -2\left[4-2f_T-(5-4f_T)\,\alpha\right]\alpha+(2f_T-1)\,(2\alpha-\eta_a)\,\eta_a \\ -2\,(\alpha-\eta_a)\left[(5-4f_T)\,\alpha-\psi_1\right]\cos\,(\psi-\psi_0)\,. \end{array} \tag{83}$$

The solution of the first order equation with homogeneous initial conditions is

$$\begin{array}{rcl} \eta_1\,(\psi) &\cong& 2\sin\left(\frac{\psi-\psi_0}{2}\right) \\ && \times\Big\{(\alpha-\eta_a)\left[\psi_1-(5-4f_T)\,\alpha\right](\psi-\psi_0)\cos\left(\frac{\psi-\psi_0}{2}\right) \\ && +\left[2\,(2f_T-4+(5-4f_T)\,\alpha)\,\alpha\right. \\ && \left.-\,(1-2f_T)\,(2\alpha-\eta_a)\,\eta_a\right]\sin\left(\frac{\psi-\psi_0}{2}\right)\Big\}\,. \end{array} \tag{84}$$

Therefore, to eliminate the secular term one must set

$$\psi_1=(5-4f_T)\,\alpha \;\underset{f_T\to 1/2}{\longrightarrow}\; 3\alpha, \tag{85}$$

and the full solution becomes

$$\begin{array}{rcl} \eta\,(\psi) &\cong& \alpha-(\alpha-\eta_a)\cos\,(\psi-\psi_0) \\ && +\epsilon\left[2\,(2f_T-4+(5-4f_T)\,\alpha)\,\alpha-(1-2f_T)\,(2\alpha-\eta_a)\,\eta_a\right] \\ && \times\left[1-\cos\,(\psi-\psi_0)\right]. \end{array} \tag{86}$$

The final conditions at the perihelion,

$$\eta(\psi_p) = \eta_p \quad \text{and} \quad \left.\frac{d\eta}{d\psi}\right|_{\psi_p} = 0, \tag{87}$$

are satisfied provided that $\psi_p = \psi_0 + \pi$ and, to first order in ϵ,

$$\alpha \cong \frac{\eta_p + \eta_a}{2} - \frac{\epsilon}{2}\left[(5 - 4f_T)\left(\eta_p^2 + \eta_a^2\right) - 2(4 - 2f_T)(\eta_p - \eta_p\eta_a + \eta_a)\right]. \tag{88}$$

The solution in terms of the original variable

$$\varphi = \psi(1 + \epsilon\psi_1 + \ldots)$$

is

$$\begin{aligned}\eta(\varphi) \;\cong\; & \alpha - (\alpha - \eta_a)\cos\left(\frac{\varphi - \varphi_0}{1 + \epsilon(5 - 4f_T)\alpha}\right) \\ & + \epsilon\left[2(2f_T - 4 + (5 - 4f_T)\alpha)\alpha - (1 - 2f_T)(2\alpha - \eta_a)\eta_a\right] \\ & \times\left[1 - \cos\left(\frac{\varphi - \varphi_0}{1 + \epsilon(5 - 4f_T)\alpha}\right)\right].\end{aligned} \tag{89}$$

The position of the perihelion after a half revolution is given in terms of the original variable by

$$\varphi_p = \varphi_0 + (1 + \epsilon\psi_1)\pi,$$

so that the precession shift $\Delta\varphi$ of the perihelion (or aphelion) for a half revolution is

$$\Delta\varphi \cong \epsilon(5 - 4f_T)\alpha\pi. \tag{90}$$

To first order in ϵ the total precession per orbit, $\sigma = 2\Delta\varphi$, is

$$\sigma \cong \frac{2\pi(5 - 4f_T)\epsilon}{1 - e^2} = \frac{2\pi(5 - 4f_T)}{1 - e^2}\frac{GM}{c^2 a}. \tag{91}$$

In terms of the non relativistic period $T_0 = 2\pi\sqrt{a^3/(GM)}$ the total precession per orbit becomes

$$\sigma \cong \frac{8\pi^3(5 - 4f_T)a^2}{c^2(1 - e^2)T_0^2} \underset{f_T\to 1/2}{\longrightarrow} \frac{24\pi^3 a^2}{c^2(1 - e^2)T_0^2}. \tag{92}$$

Introducing the "Thomas half" $f_T = 1/2$ this expression gives the correct amplitude of the perihelion shift predicted by general relativity as shown in Fig. 2. Putting $f_T = 0$ the braking action of the gravitomagnetic field is lost and the perihelion shift becomes larger by a factor of 5/3. Figure 3 illustrates this hypothetical situation.

From the Mercury data listed in Table II one obtains an orbital period $T_0 = 87.9691 \times 24 \times 3600 = 7.60053 \times 10^6$ s. Using the Sun's mass $M = 1.9891 \times 10^{30}$ kg and the gravitational constant $G = 6.67408 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ the non relativistic calculated orbital period is

$$T_0 = 2\pi\sqrt{a^3/(GM)} = 7.59937 \times 10^6\,\mathrm{s}. \tag{93}$$

With the speed of light $c = 2.99792458 \times 10^8\,\mathrm{m\,s^{-1}}$, the total precession per orbit is

$$\sigma = 6\pi GM/\left[c^2 a\left(1 - e^2\right)\right] = 5.202024 \times 10^{-7}\,\mathrm{rad}. \tag{94}$$

In one hundred Earth's years the total precession of Mercury's perihelion is

$$(100 \times 365.25636/87.9691) \times (3600 \times 180/\pi)\,\sigma = 42.9949$$

seconds of arc per century.

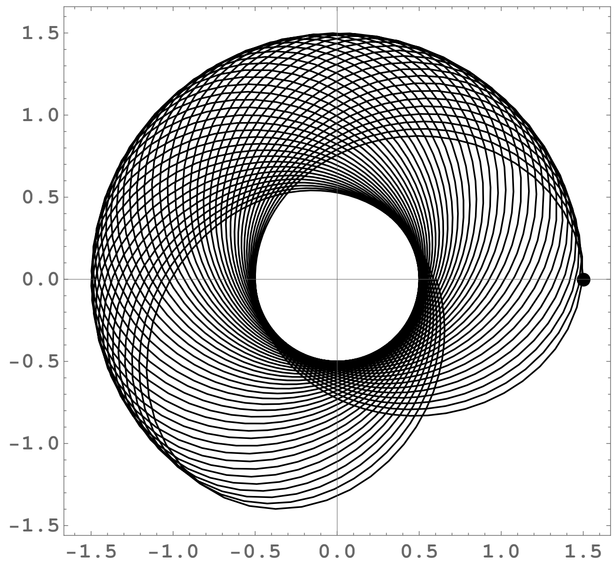

FIG. 3. Illustration of the perihelion shift for a fictitious planet in elliptical orbit in the hypothetical situation when the GM field is switched off by taking $f_T = 0$. The orbit is characterized by the same parameters shown in Fig. 2. After 50 revolutions the total perihelion shift is 244.2°, approximately 5/3 larger than the value predicted by general relativity.

Mercury data	
Semimajor orbit axis a	5.7909227×10^{10} m
Orbit eccentricity e	0.20563593
Mean orbit velocity v_0	4.7362×10^{4} m s^{-1}
Volume V	$6.0827208742\times10^{16}$ m^3
Mass m	3.30104×10^{23} kg
Length of year	87.9691 Earth days (365.25636 days/year)

TABLE II. Values taken from NASA's Solar System Exploration/Planets/Mercury website.

IV. RATE OF CHANGE OF ANGULAR MOMENTUM

Consider the time variation of the mechanical angular momentum $\boldsymbol{L} = m\gamma\boldsymbol{r}\times\boldsymbol{v}$:

$$\frac{d\boldsymbol{L}}{dt} = m\boldsymbol{r}\times\boldsymbol{v}\frac{d\gamma}{dt} + m\gamma\frac{d}{dt}\left(\boldsymbol{r}\times\boldsymbol{v}\right) = \frac{\gamma^2}{2c^2}\frac{dv^2}{dt}\boldsymbol{L} + m\gamma\boldsymbol{r}\times\frac{d\boldsymbol{v}}{dt}. \tag{95}$$

The equation of motion (20) is here repeated

$$\frac{d\boldsymbol{v}}{dt} + \frac{GM}{r^2}\widehat{\boldsymbol{r}} = \frac{GM}{r^2}\beta^2\left[\left(2f_T - 1\right)\widehat{\boldsymbol{r}} + \left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\left(4 - 2f_T\left(1+\beta^2\right)\right)\widehat{\boldsymbol{v}}\right]. \tag{96}$$

The scalar multiplication of this equation by $\boldsymbol{v}$ gives

$$\frac{\gamma^2}{2c^2}\frac{dv^2}{dt} = -\frac{GM}{c^2 r}\frac{v}{r}\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\gamma^2\left[1 - \beta^2\left(3 - 2f_T\beta^2\right)\right], \tag{97}$$

and the vector multiplication by $\boldsymbol{r}$ gives

$$\boldsymbol{r}\times\frac{d\boldsymbol{v}}{dt} = \frac{GM}{c^2 r}v^2\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\left[\left(4 - 2f_T\left(1+\beta^2\right)\right)\left(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}}\right)\right]. \tag{98}$$

Thus

$$\frac{d\boldsymbol{L}}{dt} = \frac{GM}{c^2 r}\frac{v}{r}\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\left[1 + 2\left(1 - f_T\right)\gamma^2\right]\boldsymbol{L}. \tag{99}$$

For a circular orbit with $\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}=0$ this equation gives conservation of mechanical angular momentum

$$\frac{d\boldsymbol{L}}{dt}=0. \tag{100}$$

In the general elliptical orbit case, when the planet moves toward the origin during part of the orbit, so that $v\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)<\ 0$, it follows that $d\boldsymbol{L}/dt\propto-\boldsymbol{L}$, with decreasing mechanical angular momentum.

Now, consider the evolution of the azimuthal component:

$$\frac{d\varphi}{dt}\cong\frac{\ell}{r^2}\exp\left[\frac{4GM}{c^2a}\left(1-\frac{f_T}{2}\right)\left(\frac{r-a}{r}\right)\right]. \tag{101}$$

While the circular orbit solution shows conservation of the instantaneous angular momentum due to symmetry, the elliptical orbit shows small back and forth relativistic oscillations around the semimajor axis of the orbit. The above equation shows the increase in the shift if the gravitomagnetic field is hypothetically switched off by taking $f_T=0$.

One must take into account that $\boldsymbol{L}$ corresponds to the mechanical angular momentum only, and there is a periodic exchange of angular momentum between the mechanical and gravitomagnetic field contributions to the canonical angular momentum during the orbital motion. The canonical momentum of the orbiting mass is given by

$$\boldsymbol{P}=m\left(\gamma\boldsymbol{v}+\boldsymbol{A}_g\right)=\left(1-\frac{f_T}{\gamma}\frac{GM}{c^2r}\right)\gamma m\boldsymbol{v}, \tag{102}$$

and the canonical angular momentum is defined by

$$\boldsymbol{J}=\boldsymbol{r}\times\boldsymbol{P}=\left(1-\frac{f_T}{\gamma}\frac{GM}{c^2r}\right)\boldsymbol{L}. \tag{103}$$

Thus

$$\frac{d\boldsymbol{J}}{dt}=\left(1-\frac{f_T}{\gamma}\frac{GM}{c^2r}\right)\frac{d\boldsymbol{L}}{dt}+\frac{f_T}{\gamma}\frac{GM}{c^2r}\left(\frac{1}{r}\frac{dr}{dt}-\frac{f_T}{2c^2}\frac{dv^2}{dt}\right)\boldsymbol{L}, \tag{104}$$

where the rate of change of the mechanical angular momentum is given by equation (99) and the rate of change of the squared velocity can be obtained from equation (97). Moreover, in the case of planar motion $dr/dt=v\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)$. Hence

$$\begin{aligned}\frac{d\boldsymbol{J}}{dt} &= \frac{GM}{c^2r}\frac{v}{r}\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\left(2\gamma^2+1\right)\left\{1-\frac{f_T}{\gamma}\frac{2\gamma^3-1}{2\gamma^2+1}\right.\\ &\quad\left.-\frac{GM}{c^2r}\frac{f_T}{\gamma}\left[1-f_T\left(1-\frac{\left(3-2f_T\beta^2\right)\beta^2}{2\gamma^2+1}\right)\right]\right\}\boldsymbol{L}\\ &\simeq \frac{GM}{c^2r}\frac{v}{r}\left(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}\right)\left(2\gamma^2+1\right)\left(1-\frac{f_T}{\gamma}\frac{2\gamma^3-1}{2\gamma^2+1}\right)\boldsymbol{L}+\mathcal{O}\left(G^2\right).\end{aligned} \tag{105}$$

In the conditions of planar motion the vectors $\boldsymbol{L}$, $\boldsymbol{J}$, $d\boldsymbol{L}/dt$ and $d\boldsymbol{J}/dt$ are perpendicular to the plane formed by $\widehat{\boldsymbol{r}}$ and $\widehat{\boldsymbol{v}}$. The above discussion shows that the instantaneous values of the mechanical and canonical angular momenta of the orbiting test mass are not conserved in the conditions of elliptical planar motion, only the average values are conserved.

The above discussion took into consideration only the rate of change of the test mass mechanical momentum. According to the derivation in Section II, the angular momentum of the central mass M in the frame of reference of the test mass m is related to the angular momentum of the orbiting mass m in the frame of reference of M by

$$\boldsymbol{L}_M=-\frac{M}{m\gamma_m}\boldsymbol{L}_m. \tag{106}$$

It can be easily verified that the rate of change of $\boldsymbol{L}_M$ is given by the same equation for the rate of change of $\boldsymbol{L}_m$, with the sign inverted (the vector position is inverted)

$$\frac{d\boldsymbol{L}_M}{dt}=-\frac{d\boldsymbol{L}_m}{dt}, \tag{107}$$

so that the total angular momentum of the isolated system is conserved, as expected from the conservation theorem of the angular momentum in integral form [12].

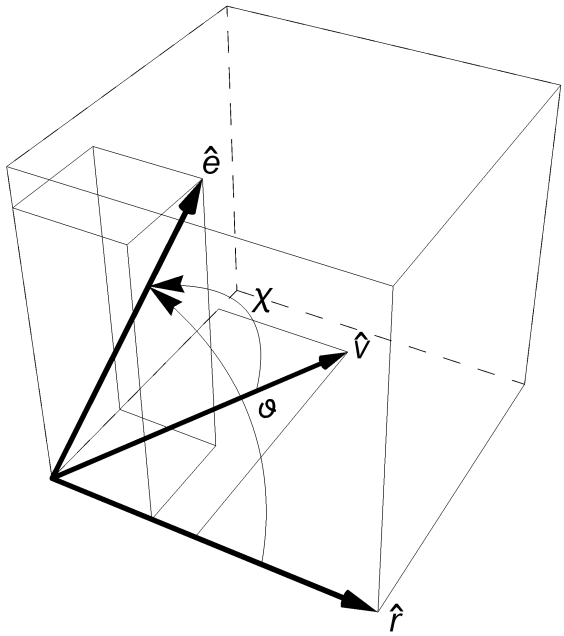

FIG. 4. Natural angle coordinates which define the three-dimensional motion of a particle.

V. COMMENTS AND CONCLUSIONS

The geodesic motion of a test particle, which does not affect the curvature of spacetime, was determined in the presence of a central body of significant mass. Of course, in the case of two significant masses, as in a binary system, the calculation can be conveniently carried out in the center of mass of the system. In general the three-dimensional motion is quite complex, depending on the initial conditions. In the present article only stable planar motions were considered, neglecting the emission of gravitational waves or the interaction with external sources. The simplest solution corresponds to circular orbits, which give a definite relation between the values of the orbital velocity and radius. It is interesting to note that a radiationless microscopic ring system (at $a \to 0$ and $\epsilon \to 0$) leads to a problem similar to the classical electrodynamics problem of an electrified point like mass with an internal structure [16]. In the weak relativistic case the classical Kepler solution is recovered.

Another class of planar, stable elliptical orbits, gives the exact results for the perihelion precession of Mercury predicted by general relativity. In both cases of circular and elliptical orbits the braking action of the gravitomagnetic field is present. The gravitomagnetic field reduces the gravitoelectric pull and opposes the test particle motion. The perihelion shift is a result of the balance between gravitoelectric and gravitomagnetic forces. Since a binary system radiates, it would be interesting to study the coalescence of the system towards the center of mass according to the extended gravitoelectromagnetic theory. The motion of a binary system is a problem that can be, presumably, tackled in the same form as the motion of the perihelion, using the Liénard-Wiechert potentials instead of the Coulomb and Biot-Savart laws. It is also intriguing how the stable circular orbits are attained, if at all. But these questions require further work.

In conclusion, the geodesic equation of motion of a test particle in the gravitational field of a massive central body was analyzed, according to the extended gravitoelectromagnetic theory, looking for stable planar orbits. The most relevant solution, which corresponds to elliptical orbits, leads to the perihelion shift predicted by general relativity.

VI. APPENDIX

In general, equation (20) describes the three-dimensional motion of a test particle. The unit vectors $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{v}}$ and $\widehat{\boldsymbol{e}}$ define a natural coordinates system. The direction $\widehat{\boldsymbol{e}}$ is defined by the pair of angles χ, subtended by $\widehat{\boldsymbol{e}}$ and $\widehat{\boldsymbol{v}} = \boldsymbol{v}/\left|\boldsymbol{v}\right|$, and ϑ, subtended by $\widehat{\boldsymbol{e}}$ and $\widehat{\boldsymbol{r}} = \boldsymbol{\nabla}\phi_g/\left|\boldsymbol{\nabla}\phi_g\right|$, respectively (Figure 4). Using the angles χ and ϑ the unit vector $\widehat{\boldsymbol{e}}$ can be written in terms of $\widehat{\boldsymbol{r}}$ and $\widehat{\boldsymbol{v}}$ in the following form

$$\widehat{\boldsymbol{e}} = e_r\widehat{\boldsymbol{r}} + e_v\widehat{\boldsymbol{v}} + e_\times\widehat{\boldsymbol{r}} \times \widehat{\boldsymbol{v}}, \tag{108}$$

where

$$e_r = \frac{\cos\vartheta - (\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\cos\chi}{1-(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})^2} \quad , \; e_v = \frac{\cos\chi - (\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\cos\vartheta}{1-(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})^2} \quad , \; e_\times = \frac{\widehat{\boldsymbol{e}}\cdot\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}}}{1-(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})^2}, \tag{109}$$

and

$$\widehat{\boldsymbol{e}}\cdot\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}} = \sqrt{1-\cos^2\chi - \cos^2\vartheta + 2(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\cos\chi\cos\vartheta - (\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})^2}. \tag{110}$$

Since $\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}} \neq 0$ in general, the system of coordinates formed by $(\widehat{\boldsymbol{r}},\widehat{\boldsymbol{v}},\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}})$ is not orthogonal. Note that $(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}})^2 = 1-(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})^2$ and $\widehat{\boldsymbol{e}}\cdot\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}} = \left[1-(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})^2\right]e_\times$. Furthermore, using the vector formula

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\,\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{C} \tag{111}$$

it can be shown that the unit vectors satisfy the following useful relations:

$$\left\{\begin{array}{rcl}
\widehat{\boldsymbol{e}}\cdot\widehat{\boldsymbol{r}} &=& \cos\vartheta \\
\widehat{\boldsymbol{e}}\cdot\widehat{\boldsymbol{v}} &=& \cos\chi \\
\widehat{\boldsymbol{e}}\times\widehat{\boldsymbol{r}} &=& -e_\times(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\,\widehat{\boldsymbol{r}} + e_\times\widehat{\boldsymbol{v}} - e_v(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}}) \\
\widehat{\boldsymbol{e}}\times\widehat{\boldsymbol{v}} &=& -e_\times\widehat{\boldsymbol{r}} + e_\times(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\,\widehat{\boldsymbol{v}} + e_r(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}}) \\
\widehat{\boldsymbol{e}}\times(\widehat{\boldsymbol{e}}\times\widehat{\boldsymbol{v}}) &=& (\widehat{\boldsymbol{e}}\cdot\widehat{\boldsymbol{v}})\,\widehat{\boldsymbol{e}} - \widehat{\boldsymbol{v}} = e_r\cos\chi\widehat{\boldsymbol{r}} + (e_v\cos\chi - 1)\,\widehat{\boldsymbol{v}} + e_\times\cos\chi\,(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}}) \\
\widehat{\boldsymbol{r}}\cdot[\widehat{\boldsymbol{e}}\times(\widehat{\boldsymbol{e}}\times\widehat{\boldsymbol{v}})] &=& e_r\cos\chi + (e_v\cos\chi - 1)\,(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}) \\
\widehat{\boldsymbol{r}}\times[\widehat{\boldsymbol{e}}\times(\widehat{\boldsymbol{e}}\times\widehat{\boldsymbol{v}})] &=& e_\times\cos\chi\,(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})\,\widehat{\boldsymbol{r}} - e_\times\cos\chi\widehat{\boldsymbol{v}} - (1-e_v\cos\chi)\,(\widehat{\boldsymbol{r}}\times\widehat{\boldsymbol{v}})
\end{array}\right. \tag{112}$$

Also

$$\left\{\begin{array}{rcl}
e_r + e_v(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}) &=& \cos\vartheta \\
e_r(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}}) + e_v &=& \cos\chi \\
e_r^2 + e_\times^2 &=& \dfrac{1-\cos^2\chi}{1-(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{v}})^2} \\
\widehat{\boldsymbol{e}}\cdot\widehat{\boldsymbol{e}} &=& 1
\end{array}\right. \tag{113}$$

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